Endogenous Extreme Events and the Dual Role of Prices*

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Abstract

Extreme events in financial markets are often generated by shocks that are generated within the system, rather than those that arrive from outside the system. The combination of risk-sensitive behavior rules and the coordinated actions implied by mark-to-market accounting can result in outcome distributions with fat tails, even if the fundamental shocks are Gaussian. We illustrate such "endogenous extreme events" through the pricing density resulting from dynamic hedging of options and the "flash crash" of May 2010.

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1 Introduction

The global financial crisis of 2007-9 and the market turmoil that accompanied it have renewed interest in understanding the nature and consequence of extreme events. Financial crises are often characterized by large price changes, but large price changes by themselves need not constitute a crisis. Public announcements of important macroeconomic statistics, such as the monthly U.S. employment report, are sometimes marked by large, discrete price changes at the time of announcement. However, such price changes are arguably the signs of a smoothly functioning market that is able to incorporate new information quickly. The market typically finds composure quite rapidly after such discrete price changes as the new information is absorbed by the market.

Instead, we are interested in episodes where shocks are amplified by the actions of the economic agents themselves. Rather like a tropical storm over a warm sea, crisis episodes appear to gather more energy as they develop. As financial conditions worsen, the willingness of market participants to bear risk seemingly evaporates. The global financial crisis that erupted in the summer of 2007 served as a laboratory for many such distress episodes.

Our objectives are somewhat different from the related asset pricing inquiry that has asked whether "rare disasters" can account for the risk premium puzzle in asset prices or the returns associated with carry trades.[1] Rather than asking whether the prices prior to the crisis can be rationalized, we address the crisis dynamic itself. Although we do not address the asset pricing consequences directly, our discussion complements the asset pricing inquiry by airing the possible mechanisms that may account for such extreme events. Indeed, our contribution is to show that "rare disasters" are often man-made rather than acts of Nature.

Our approach also differs from the statistical approach typified by Extreme Value Theory (EVT). Economists have long recognized that the Gaus-

[1] Rietz (1988), Barro (2006) and Weitzman (2007) address the risk premium in asset prices through the lens of rare disasters. Fahri and Gabaix (2010) argue that the possibility of rare disasters can account for the excess returns associated with currency carry trades. In a series of papers, Burnside, Eichenbaum, Kleshchelski and Rebelo (2006, 2007, 2008) have explored the extent to which conventional asset pricing models can explain the returns to carry trade positions, and point to the importance of rare jumps in the stochastic discount factor itself - a form of peso problem. Plantin and Shin (2010) model the "up by the stairs, down by the elevator" price dynamics of carry trade currencies.

sian distribution is not sufficient for describing economic variables, since at least the work of Vilfredo Pareto (Pareto (1898)). Pareto blazed the trail on the study of "fat tails" of probability densities based on the concept of power laws. See, for example, Embrechts et al. (1996) for a survey of EVT and power laws.

While Pareto applied his research to income distributions, such analysis equally applies to returns and financial assets. Mandelbrot (1963) and Fama (1965) showed that financial returns exhibit fat tails, with Jansen and de Vries (1991) the first to apply EVT to finance. Since then we have seen a large number of studies.

EVT provides many useful insights on extreme market outcomes. However, there are two key factors that limit the application of EVT in finance. First, it only applies relatively far out in the tails, generally for events with probability much less than 1%, and it can be quite challenging identifying where exactly it applies. Secondly, it assumes the underlying data is identically and independently distributed, or that the tails exhibit a restricted form of dependence. If the underlying data is subject to apparent structural breaks, EVT becomes less relevant. This is exactly a feature of financial returns.

Our approach is different, and emphasizes the man-made nature of extreme events. The main theme of our paper can be encapsulated in terms of the *dual role of prices*. By "dual role", we mean that prices not only reflect the underlying economic fundamentals, they are also an imperative to action. That is, prices induce actions on the part of the economic agents. If some actions are the consequence of binding constraints and exert harmful spillover effects on others, then price changes can bring about amplifying spillover effects that disrupt the smooth working of the market, and sometimes shut down the market completely. Financial crises could almost be defined as episodes where the allocational role of prices break down. The action-inducing role of price changes introduce distortions and cause an amplified spiral of price changes and actions that can cause great damage along the way.

In order to motivate our discussion, it is useful to begin with an example from outside economics and examine the case of the Millennium Bridge, first discussed in Danielsson and Shin (2003). The discussion below draws on Shin (2010).

Lessons from the Millennium Bridge

The Millennium Bridge in London was constructed as part of the Millennium celebrations in the year 2000. It was the first new crossing over the River Thames for over a hundred years. The sleek 325 metre-long structure used an innovative "lateral suspension" design, built without the tall supporting columns that are more familiar with other suspension bridges. The bridge was opened by the Queen on a sunny day in June, and the press was there in force. Many thousands of people turned up after the tape was cut and crowded on to the bridge to savor the occasion. However, within moments of the bridge's opening, it began to shake violently. The shaking was so severe that many pedestrians clung on to the side-rails, as shown in video news clips of the opening day.[2] The bridge was closed shortly after the opening and was to remain closed for 18 months.

When engineers used shaking machines to send vibrations through the bridge, they found that horizontal shaking at 1 hertz (that is, at one cycle per second) set off the wobble seen on the opening day. Now, this was an important clue, since normal walking pace is around two strides per second, which means that we're on our left foot every second and on our right foot every second. Walking produces a vertical force (depending on our body mass) of around 750 Newtons or 165 pounds at 2 hertz. However, there is also a small sideways force caused by the sway of our body mass due to the fact that our legs are slightly apart. Anyone who has been on a rope bridge should be well aware of the existence of this sideways force. This force (around 25 Newtons or 5.5 pounds) is directed to the left when we are on our left foot, and to the right when we are on our right foot. This force occurs at half the frequency (or at 1 hertz). This was the frequency that was causing the problems.

But why should this be a problem? We know that soldiers should break step before crossing a bridge. For thousands of pedestrians walking at random, one person's sway to the left should be cancelled out by another's sway to the right. If anything, the principle of diversification suggests that having lots of people on the bridge is the best way of cancelling out the sideways forces on the bridge.

Or, to put it another way, what is the probability that a thousand people walking at random will end up walking exactly in step, and remain in

[2] http://news.bbc.co.uk/hi/english/static/in_depth/uk/2000/millennium_bridge/default.stm. See also the youtube video on http://www.youtube.com/watch?v=eAXVa__XWZ8

lock-step thereafter? It is tempting to say "close to zero". After all, if each person's step is an independent event, then the probability of everyone walking in step would be the product of many small numbers - giving us a probability close to zero.

However, we have to take into account the way that people react to their environment. Pedestrians on the bridge react to how the bridge is moving. When the bridge moves from under your feet, it is a natural reaction to adjust your stance to regain balance. But here is the catch. When the bridge moves, everyone adjusts his or her stance *at the same time*. This synchronised movement pushes the bridge that the people are standing on, and makes the bridge move even more. This, in turn, makes the pedestrians adjust their stance more drastically, and so on. In other words, the wobble of the bridge feeds on itself. When the bridge wobbles, everyone adjusts their stance, which makes the wobble even worse. So, the wobble will continue and get stronger even though the initial shock (say, a small gust of wind) has long passed.

Arup, the bridge's engineers found that the critical threshold for the number of pedestrians that started the wobble was 156. Up to that number, the movement increased only slightly as more people came on the bridge. However, with ten more people, the wobble increased at a sharply higher rate.[3] The wobble is an example of a shock that is generated and amplified *within* the system. It is very different from a shock that comes from a storm or an earthquake which come from outside the system. Stress testing on the computer that looks only at storms, earthquakes and heavy loads on the bridge would regard the events on the opening day as a "perfect storm". But this is a perfect storm that is guaranteed to come every day.

What does all this have to do with financial markets? Financial markets are the supreme example of an environment where individuals react to what's happening around them, and where individuals' actions affect the outcomes themselves. The pedestrians on the Millennium Bridge are like modern banks that react to price changes, and the movements in the bridge itself are rather like price changes in the market. So, under the right conditions, price changes will elicit reactions from the banks, which move prices, which elicit further reactions, and so on.

The Millennium Bridge analogy serves to highlight the dual role of prices.

[3] http://www.arup.com/millenniumbridge/challenge/results.html. See also http://www.youtube.com/watch?v=eAXVa__XWZ8

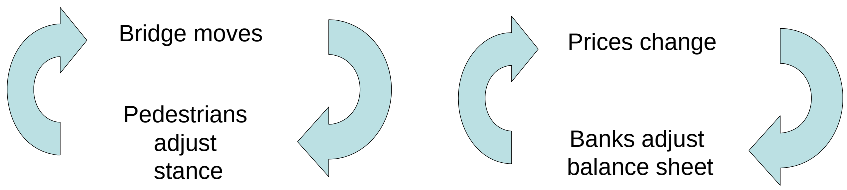

Figure 1. Feedback in Financial Systems (Source: Shin (2010))

Not only are prices a reflection of the underlying economic fundamentals, they are also an imperative to action. That is, prices induce actions on the part of the economic agents. When actions are the result of binding constraints and exert harmful spillover effects on others, then the double-edged nature of prices exerts its biggest effect. The problem comes when the *reliance* on market prices *distorts* those same market prices. The more weight is given to prices in making decisions, the greater are the spillover effects that ultimately undermine the integrity of those prices. When prices are so distorted, their allocational role is severely impaired. Far from promoting efficiency, contaminated prices undermine their allocational role.

Flash Crash of May 2010

The non-linear effects of the endogenous risk type plausibly arise in algorithmic trading environments, which have emerged as a central issue in market microstructure and the regulation of exchanges and trading platforms.

Algorithmic trading either has actions hard-coded into their programs that directly lead to positive feedbacks, or the algorithms do not have such a behaviour coded directly into the programmes but some higher level interventions by the controlling or the supervising entity effectively can decide to overrule the algorithm, and thereby create the feedbacks.

As an example, consider the "flash-crash" epsiode of May 6th 2010, when the US stock market was buffetted by unprecedented turbulence in a short period in the afternoon of May 6th. SEC (2010) is the official report on the episode from the US Securities and Exchange Commission. Irrespective of whether or not the official version of the events of May 6th 2010 turns out to be accurate and complete, it outlines a possible scenario whereby some algorithms may *directly* create feedback effects due to a lack of common sense in the coding.

In a nutshell, a simplified scenario would be as follows. An execution algorithm by a investing firm is directed to sell a large number of securities. The algorithm gauges the market impact it may have by looking at the market volume and is instructed to sell more if the volume is higher. Whereas in many market-sensitive trading stratgies it is prices that play the dual roles of gauges of value as well as of imperatives to act, in this case volumes play the dual role of indicators of intensity of transactions as well as the role of imperatives to be followed.

The original large sale finds buyers, most likely high-frequency traders (HFTs) given their speed advantage. If it turns out that there are no new real-money investors stepping in, the algorithms of the lightly capitalized HFTs that act as market makers may pass the securities around like a hot potato, generating more volume. This volume then teases the original algorithm to sell even more, closing the feedback loop until far out-of-the-money limit orders are hit and the order books are emptied. The destabilizing feedback loop in this instance has been brought about through the interaction of two distinct algorithms. Diagrammatically,

$$\begin{array}{ccc} \text{general algo selling} & \Rightarrow & \text{HFTs pass the parcel, wait for real money investor} \\ \Uparrow & & \Downarrow \\ \text{execution algo sells more if volume is up} & \Leftarrow & \text{volume shoots up} \end{array}$$

There are many more such sources of endogenous risk in computer-based trading, see for instance Shin and Zigrand (2011).

So far, the analogy between the Millennium Bridge example and the financial market has been informal. In order to illustrate the effects in more detail, we will now examine a specific case of endogenous amplification in financial markets due to the dynamic hedging of options. Dynamic hedging in general refers to the practice of active adjustment one's portfolio so as to leave the portfolio hedged against future shocks. The specific case we examine is the dynamic hedging of options, drawing on the discussion in Danielsson and Shin (2003) and Shin (2010).

2 Dynamic Hedging of Options

In the 1980s, specialized fund managers put into practice the principles that underpin the Black and Scholes (1973) model for option pricing and set up funds that became known as portfolio insurers. Bookstaber (2007) gives a first-hand description from a practitioner on the scene at the time.

Portfolio insurance attemps to replicate the payoffs that arise from holding a put option by trading actively in the market. A put option gives the holder the right (but not the obligation) to sell a particular asset at a pre-agreed price (the *strike price*, or *exercise price*) at a particular date in the future (the *expiry date*). At the expiry date, the value of the put option is large when the price of the underlying asset is far below the exercise price, since the holder of the put can buy the underlying asset at the low prevailing market price, and then sell it at the higher exercise price, pocketing the difference. When the price of the underlying asset is *above* the exercise price, the holder of the option will not exercise the option, and the option will expire worthless.

Before the expiry date, the option has a value that is above its value at expiry, since the price S of the underlying asset at expiry is uncertain. Even if the current price S is above the strike price X, there is a chance that S will drift lower below X before expiry. As long as this is a possibility, the option has a positive value. The farther in the future is the expiry date, the greater is the uncertainty, and the greater is the value of option for any given price of the underlying asset today.[4] The value of a put option is increasing when the price S falls. Also, *the rate* which the option increases in value is itself increasing as S falls.

Replicating the payoff of a put option through dynamic hedging attempts to position one's portfolio in reaction to price changes in order to mimic the payoffs from a put option at expiry. There are two requirements. Since a put option pays out more when price is low, this means maintaining a short position in the underlying asset. Since the slope of the put option's value becomes steeper as the price falls, this means taking an even larger short position when the underlying asset falls in price. In other words, dynamic hedging dictates that when the price *falls*, you *sell* more of the asset. Replicating a put option through dynamic trading entails a "sell cheap, buy dear" strategy.

Why might it make sense to replicate a put option, rather than just buying a put option? Options that trade in organized exchanges are limited to certain well-established markets, and only for relatively short expiry dates. For very long-dated options, or for specific portfolios, dynamic replication may be the only avenue open to an investor if he/she wishes to attempt to

[4] This may not be true if the risk-free interest rate is very high, but we assume for the sake of simplicity that the risk-free rate is zero for the rest of the section.

hedge the value of an investment holding. One could approach one of the large banks or securities firms and ask it to sell an option to you. But you will need to pay for the privilege of buying the option. For instance, a fund manager who has sold long-term retail funds that guarantee the initial investment, the implicit put must be replicated in some way.

For the bank that sells you the option, it is incurring the liabilities generated by having sold the option. For this reason, even if the bank sells you an over-the-counter (OTC) option tailored to your needs, this does not mean that dynamic hedging becomes irrelevant. Once the bank sells the option to you, the bank is holding a risky liability, and will want to hedge this risk. The burden of replication is placed on the bank that has sold the option. So, as long as some party has to bear the risk of the liabilities generated by the option, dynamic hedging becomes relevant.

2.1 Delta Hedging

In its simplest form, dynamic hedging relies on the *delta* of the option. To fix ideas, focus on the task of replicating the payoff of a put option. The delta of a put option is the rate of change of the put option price with respect to the change in the price of the underlying asset. Thus, if P is the price of the put option and S is the price of the underlying asset, the delta Δ is given by $\Delta = dP/dS$. For a put option, its delta lies between -1 and 0. Black and Scholes in their famous paper on option pricing noted that the portfolio consisting of:

$$\left\{ \begin{array}{cc} \Delta & \text{underlying asset} \\ -1 & \text{put option} \end{array} \right.$$

is locally risk-free with respect to changes in S. This is because when the price changes slightly, the gain from the holding of the underlying asset (given by Δ) is matched by an exactly offsetting loss in the price of the put option ($-\Delta$). This insight is used in the derivation of the Black-Scholes formula by arguing that the above portfolio must earn same return as the risk free asset.

The delta of a put option can be pictured in Figure 2. The delta is the slope of P with respect to S, and hence lies below the horizontal axis. The delta goes to -1 as the price of the underlying security S falls, and tends to 0 as the price of the underlying security increases. As time progresses to the expiry of the option, the price of the option gets closer to the kinked curve

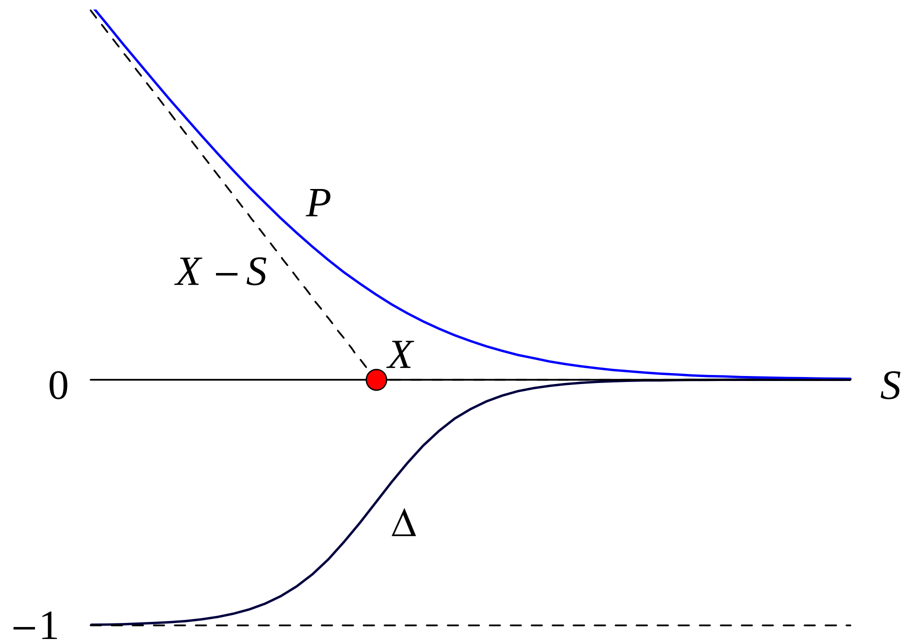

Figure 2. Delta of Put Option

with the kink at the exercise price X. So, the delta behaves more like the step function that jumps from -1 to 0 at exercise price X.

At expiry, there are two possible values of delta. If the option expires "in the money" so that $S < X$, then we are on the negatively sloped part of the curve so that $\Delta = -1$. However, if the option expires "out of the money", we are on the flat part of the curve so that $\Delta = 0$.

The payoff from the put option can be replicated by holding a suitable portfolio of the underlying asset and cash, and adjusting the position over time in response to realised outcomes. Suppose a trader starts with a cash balance of P, and suppose that P is also the market price of the put option that the trader wishes to replicate. With this wealth, the trader can either purchase the put option itself, or purchase the portfolio given by:

$$\left\{ \begin{array}{cc} \Delta & \text{underlying asset} \\ -S\Delta + P & \text{cash} \end{array} \right. \tag{1}$$

The value of this portfolio is also P, since the Δ units of the underlying asset has price $-S\Delta$. Remember that Δ is negative since the trader wishes to replicate a put option. The portfolio given by (1) is financed by selling short

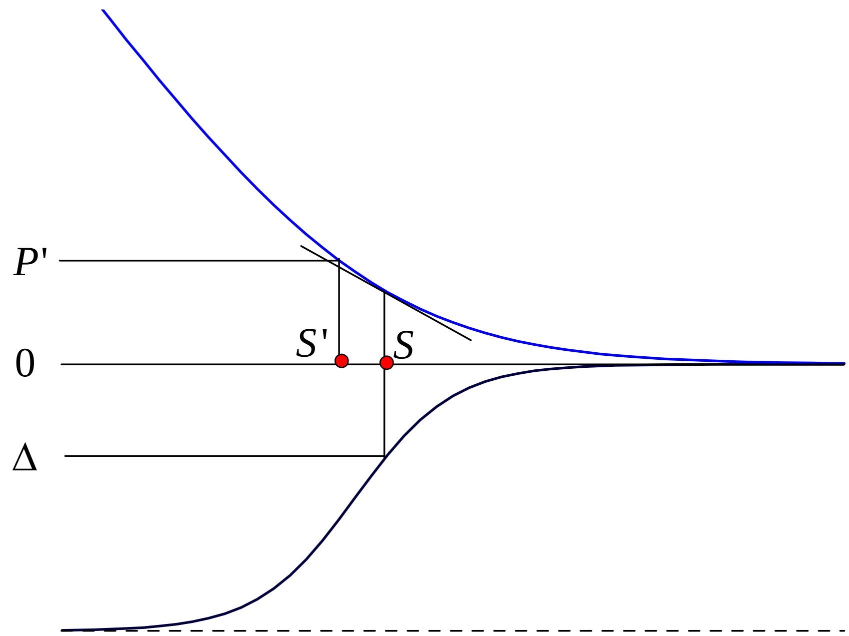

Figure 3. Delta following price change

$|\Delta|$ units of underlying asset at price S, and adding the proceeds to the cash balance.

Now, suppose price changes to S'. The value of the portfolio at the new price is

$$\begin{aligned} & \overbrace{\Delta \cdot S'}^{\text{short asset}} + \overbrace{P - S\Delta}^{\text{cash}} \\ = \quad & P + \Delta\left(S' - S\right) \\ \simeq \quad & P' \end{aligned}$$

where P' is the price of the put option given price S'. Figure 3 illustrates the change in the price of the portfolio following the price change, and how it relates to the shift in the price of the put option itself. The trader manages to approximate the wealth of a trader who starts out by holding the put option itself, in the sense that the trader's portfolio value moves along the tangency line at the old price S. Since the approximation is linear, the accuracy of the approximation is greater the smaller is the price change.

After the price change, the trader can repeat his procedure at the new price S'. At the new price S', the investor forms the new portfolio:

$$\left\{ \begin{array}{cc} \Delta' & \text{underlying asset} \\ -S'\Delta' + P' & \text{cash} \end{array} \right. \tag{2}$$

which is affordable (approximately) given his new wealth of P'. Suppose that the trader repeats this procedure of forming the new portfolio in response to price changes so that he maintains a position of Δ in the underlying security, and where the cash position adjusts as a result of the shift in the portfolio. When the price falls, the delta becomes more negative, meaning that the trader sells more of the underlying security, and thereby adding to the cash balance by the amount of the dollar value of securities sold short in that round. Conversely, if the price rises, then the delta becomes less negative, meaning that the trader has to buy back some of the security, thereby dipping into his cash balance to make the purchase. The cash balance will adjust in this way as a result of new sales and purchases.

Proceeding in this way, let us suppose that the trader reaches the expiry date of the option. There are two cases we need to consider, depending on whether the option expires in the money or out of the money. If the option expires in the money (i.e. when the price S is below the exercise price X), we have $\Delta = -1$, so that the portfolio given by (2) is

$$\left\{ \begin{array}{cc} -1 & \text{underlying asset} \\ S + (X - S) & \text{cash} \end{array} \right.$$

In this case, the trader has a balance sheet in which he has cash of $S+(X-S)$ on the asset side, and 1 unit of the underlying security on the liabilities side. The difference between the two is the equity of the trader. Since the price of the underlying security is S, the value of equity is

$$\underbrace{S + (X - S)}_{\text{asset}} - \underbrace{S}_{\text{liability}} = X - S \tag{3}$$

Another way to think about this is to imagine the trader buying back the one unit of the security at expiry, at the price of S. With a cash balance of $S + (X - S)$, paying out S leaves the trader with $X - S$.

The second case is when the option expires out of the money. In this case, the price of the underlying security S lies above the exercise price X. So, the portfolio (2) takes the particular simple form:

$$\left\{ \begin{array}{cc} 0 & \text{underlying asset} \\ 0 & \text{cash} \end{array} \right. \tag{4}$$

In this case, the equity of the trader is zero. So, taking account of the two possible cases taken by the trader's portfolio at the expiration date, the final value of the trader's portfolio is the larger of $X-S$ and zero. In other words, the payoff at expiry of the trader who follows the strategy of keeping a delta position in the underlying security is given by

$$\max\{X-S,0\}$$

But this payoff is exactly the payoff achieved by the alternative strategy for the trader in which he pays P to buy one unit of the put option, and holds it to expiry. In this way, the strategy of hedging by holding a delta position in the underlying security enables the trader to mimic the payoff of buying a put option and holding it.

2.2 Numerical Example

Let us first examine a numerical example for the case where returns are given exogenously - that is, unaffected by the actions of the traders. This is the case made famous by Black and Scholes (1973) and examined in textbooks, such as Hull (2009). Suppose the initial price of the underlying security is 100. A trader wishes to replicate the payoff of the put option with strike price 90 by rebalancing his portfolio at the end of each week. We suppose that the trader starts with a zero cash balance, but can borrow and lend at some risk-free rate r. The option expires in 20 weeks. Suppose also that the process governing the evolution of the security's price is such that the Black-Scholes (1973) option pricing formula is valid.

According to the Black-Scholes formula for option pricing, the delta of a put option at time t is given by

$$N(d_1)-1$$

where $N(.)$ is cumulative distribution function of the standard normal, and d_1 is given by

$$d_1=\frac{\ln\left(\frac{S}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

where X is the exercise price of the option, S is the price of underlying asset, r is the risk-free interest rate and T is the expiry date of the option and σ is

Table 1. No Feedback, In the Money

Week	T - t	Random	Price	log(p/x)	d_1	Delta	Purchases	Cash Flow	Cash Stock
0	0.385		100.000	0.105	0.757	-0.224	-0.224	22.450	22.450
1	0.365	-0.026	97.378	0.079	0.597	-0.275	-0.051	4.944	27.394
2	0.346	0.014	98.738	0.093	0.704	-0.241	0.034	-3.397	23.997
3	0.327	-0.046	94.213	0.046	0.392	-0.348	-0.107	10.065	34.062
4	0.308	0.015	95.617	0.061	0.506	-0.306	0.041	-3.944	30.118
5	0.288	0.006	96.144	0.066	0.559	-0.288	0.018	-1.764	28.353
6	0.269	0.021	98.138	0.087	0.732	-0.232	0.056	-5.502	22.851
7	0.250	-0.047	93.485	0.038	0.366	-0.357	-0.125	11.684	34.535
8	0.231	0.024	95.685	0.061	0.570	-0.284	0.073	-6.955	27.580
9	0.212	-0.037	92.180	0.024	0.266	-0.395	-0.111	10.227	37.807
10	0.192	-0.010	91.283	0.014	0.184	-0.427	-0.032	2.901	40.708
11	0.173	-0.048	86.910	-0.035	-0.284	-0.612	-0.185	16.054	56.762
12	0.154	-0.045	83.001	-0.081	-0.777	-0.781	-0.170	14.072	70.834
13	0.135	-0.037	79.944	-0.118	-1.246	-0.894	-0.112	8.977	79.811
14	0.115	-0.043	76.546	-0.162	-1.864	-0.969	-0.075	5.761	85.572
15	0.096	-0.020	75.005	-0.182	-2.312	-0.990	-0.021	1.557	87.129
16	0.077	0.044	78.313	-0.139	-1.971	-0.976	0.014	-1.093	86.036
17	0.058	-0.036	75.531	-0.175	-2.889	-0.998	-0.022	1.692	87.728
18	0.038	-0.047	71.989	-0.223	-4.530	-1.000	-0.002	0.139	87.867
19	0.019	0.011	72.788	-0.212	-6.105	-1.000	0.000	0.000	87.867
20	0.000	-0.035	70.236	-0.248		-1.000	0.000	0.000	87.867

standard deviation of return of the underlying security. The Black-Scholes formula for the price for the put at t is

$$P = Xe^{-r(T-t)}N\left(-d_2\right) + S\left(N\left(d_1\right) - 1\right)$$

where $d_2 = d_1 - \sigma\sqrt{T-t}$. For economy of notation, set $r = 0$. Quantities such as the time to expiry $T - t$ and volatility σ will be measured in units of years. In the case we want to examine, there are 20 weeks to expiry, so that the time to expiry at the initial date is 0.3846 years. The exogenous returns are drawn from a normal density for weekly returns that is consistent with a 25% yearly volatility σ. The returns each week are assumed to be independent. The annual standard deviation is converted to a weekly standard deviation by dividing by the square root of 52, the number of weeks in a year.

Table 1 shows draws where the returns are unfavorable to the security value so that the put option ends up in the money. Initially, the price is 100, and the trader begins with a short position in the underlying security of -0.2245. However, the return in the first week is -2.6%, lowering the price to 97.38. The delta becomes more negative, at -0.2753, which is met by the trader selling additional units of the risky security, and adding 4.94 to the cash balance. At the end of the first week, the trader has a cash balance of 27.39, as seen in the last column.

Proceeding in this way, the trader adjusts his portfolio at the end of each week in response to the weekly realised return. The random draws push the security price down, so that at the end of the 20 weeks, the security price ends up at 70.24. The option ends up in the money, and the delta goes to -1 rapidly in the last few weeks. The cash balance at the end of the 20 weeks is 87.87.

At the end of the 20 weeks, the trader has a portfolio consisting of a cash balance of 87.87 and a liability of one unit of the risky security. Since the price of the security is 70.24 at that date, the equity of the trader is given by

$$87.87 - 70.24 = 17.63$$

Having started off with a zero cash balance, 17.63 is the net gain from having replicated the put option. We can compare this outcome to the alternative that was open to the trader of buying one unit of the put option and then waiting for the expiry of the option at the end of 20 weeks. The Black-Scholes price of the option at date 0 with strike price 90 is 2.17. Meanwhile, the option ends up in the money by the difference between 90 and 70.24. Hence, the net gain to the trader is

$$90 - 70.24 - 2.17 = 17.59$$

which is very close to the 17.63 that is made by the trader who uses delta hedging to replicate the put option. In this particular numerical example, the outcome of the delta hedging is extremely close to the outcome given by buying and holding the put.

Delta hedging rests on being able to sell the security when the price falls, and buying the security when its price rises. In other words, it is a strategy that chases price moves up or down. The strategy rests on there being someone who buys when you want to sell. However, when there is feedback from the actions of traders to the price moves seen on the market, then there is the potential for amplified responses, where price falls elicit more selling, which pushes price down, which then elicits further selling. When the conditions are ripe (on which more below), delta hedging can generate a price spiral where selling and market dynamics create a feedback loop.

To illustrate such a possibility, let us examine a slightly modified version of the example with a price feedback effect where sales and purchases impact on price changes in the market. The idea is that selling creates downward pressure on price and buying creates an upward pressure on price.

Table 2. With Feedback, In the Money

Week	T - t	Random	Theoretical Price	Actual Price	Delta	Purchases	Cash Flow	Cash Stock
0	0.385		100.000	100.000	-0.224	-0.224	22.450	22.450
1	0.365	-0.026	97.378	97.378	-0.275	-0.051	4.944	27.394
2	0.346	0.014	98.738	93.793	-0.362	-0.086	8.095	35.490
3	0.327	-0.046	94.213	81.400	-0.736	-0.374	30.480	65.969
4	0.308	0.015	95.617	52.133	-1.000	-0.264	13.759	79.728
5	0.288	0.006	96.144	38.662	-1.000	0.000	0.002	79.730
6	0.269	0.021	98.138	39.461	-1.000	0.000	0.000	79.730
7	0.250	-0.047	93.485	37.591	-1.000	0.000	0.000	79.730
8	0.231	0.024	95.685	38.475	-1.000	0.000	0.000	79.730
9	0.212	-0.037	92.180	37.066	-1.000	0.000	0.000	79.730
10	0.192	-0.010	91.283	36.705	-1.000	0.000	0.000	79.730
11	0.173	-0.048	86.910	34.946	-1.000	0.000	0.000	79.730
12	0.154	-0.045	83.001	33.375	-1.000	0.000	0.000	79.730
13	0.135	-0.037	79.944	32.145	-1.000	0.000	0.000	79.730
14	0.115	-0.043	76.546	30.779	-1.000	0.000	0.000	79.730
15	0.096	-0.020	75.005	30.160	-1.000	0.000	0.000	79.730
16	0.077	0.044	78.313	31.490	-1.000	0.000	0.000	79.730
17	0.058	-0.036	75.531	30.371	-1.000	0.000	0.000	79.730
18	0.038	-0.047	71.989	28.947	-1.000	0.000	0.000	79.730
19	0.019	0.011	72.788	29.268	-1.000	0.000	0.000	79.730
20	0.000	-0.035	70.236	28.242	-1.000	0.000	0.000	79.730

For concreteness, first consider the case where the realized return from date $t-1$ to t is given by

$$1 + r_t + y_t \tag{5}$$

where r_t is the exogenous random return given in the third column of the tables examined above and y_t is the *purchase* of the security as given by the column in the tables labelled as "Purchases". This is the purchase dictated by delta hedging, where the portfolio is required to be rebalanced after the price change to reflect the new value of the optional delta. Since the trader maintains a position in the security of delta of the option, the "Purchases" column reflects the change in the delta from one date to the next.

Table 2 tracks the outcome over time. There are now two columns for the price sequence. First there is a "Theoretical Price" column that reflects just the exogenous returns $\{r_t\}$. But the column marked "Actual Price" incorporates the selling and buying pressure y_t also. At date 0 the starting price of the security is 100, the delta is -0.2245, so that the trader's portfolio at the end of date 0 consists of a short position of 0.2245 units of the security and a cash balance of 22.45. At the end of week 1, the fundamental return is -2.26%, which drives down the price to 97.38, as before in Table 1.

However, this is when the downward spiral begins to gather momentum. The sale at the end of date 1 feeds into the return for week 2. The "fundamental" return in week 2 is positive, namely 1.4%. However, this positive fundamental return is swamped by the downward pressure on prices exerted by the sale of 0.0508 units of the security at the end of week 1. The realized return that combines the fundamental shock and the downward pressure on price from sales is given by

$$0.014 - 0.0508 = -0.0368$$

so that the actual price at the end of week 2 is given by

$$97.38 \times (1 - 0.0368) = 93.79$$

This compares with the theoretical price of 98.74 that takes account only of the exogenous return. The potency of the feedback effect then takes a grip on the price process. With each massive sale in one period, the return in the subsequent period is depressed, which generates more sales, and so on.

The upshot of the feedback is clear from the price path in Table 2. The price falls very rapidly from the starting price of 100. By the end of week 4, the price has crashed to 52.13, compared to the theoretical price of 95.62. The column tracking the delta of the option reflects the rapid price decline. By the end of week 4, the delta has in effect reached its lower bound of -1. Once the delta reaches -1, the price of the security remains deep in the money, and so the delta remains at -1 until the expiry of the option. Since there is no further change in the delta, there is no trading of the security either. Figure 4 plots the price paths with and without feedback for the case where the option ends up in the money.

At expiry, the security's actual price has crashed to 28.24. The cash balance of the delta-hedging trader stands at 79.73. Since the trader has a liability of 1 unit of the security, the equity of the trader is

$$79.73 - 28.24 = 51.49$$

Had the trader bought the put option at date 0 at the Black-Scholes price of 2.17, the net position at the time of expiry would have been

$$90 - 28.24 - 2.17 = 59.59$$

which is substantially larger than the outcome of the delta hedging. Again, this is an illustration that when there is feedback, the Black-Scholes formula is underpricing the put option.

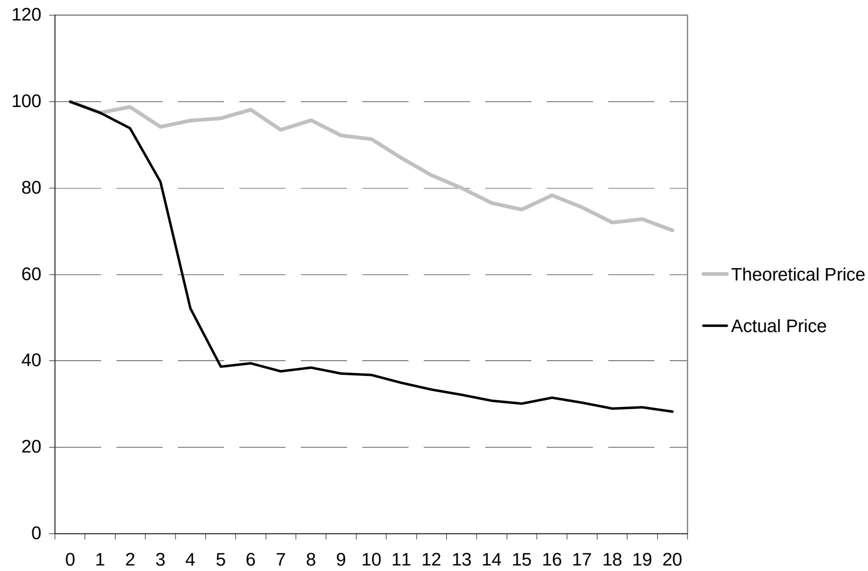

Figure 4. Price paths with and without feedback, out of the money case

3 Numerical Simulation of Dynamic Case

Building on the simple example of feedback avove, we now investigate more systematically the potential underpricing of the option when feedback effects are neglected.

We will proceed by developing the simple example above by incorporating not only the price pressure generated by sales and purchases, but we make the more realistic assumption that the market price reverts back to some fundamental value after the price shock due to a sale or purchase. In other words, the price shock due to a sale or purchase is only temporary. Kyle (1985) has popularized the concept of "resiliency" of the market to describe such a reversion to the fundamental price.

In addition, we will examine the case where the strike price of the option to be hedged also shifts, in line with the current market price. This feature is designed to capture the idea that the hedging strategies put in place by market participants will closely mirror the current prevailing market price.

Specifically, the simulation is set up as follows. The option expires at date T, and the remaining time to expiry is $T-t$. Time is measured in units of one year, as before. At the beginning, the agent decides to dynamically replicate a put option, with strike price X set at fraction ϕ of yesterday's price, i.e., $X = \phi P$, rebalancing every $dt = 1/365$ years, so the agent re–balances everyday. This means that the agent starts out being fully delta-hedged. The number of days when the agent does this is denoted by N, so at the beginning of the period, $T - t = Ndt$. The annual volatility of returns of the underlying asset is indicated by σ.

On the day after the option expires, the agent may decide to repeat the experience. We indicate the number of times agent does this by Q.

The sequence of agents actions is as follows. We indicate the days by i. Then the sequencing in other numerical simulation is given as follows.

day 1 Price, P_1 is realized

day 2
1. The strike price X adjusts to $X = \phi P_1$
2. Calculate the option delta Δ_2
3. At all times the agent maintains delta position Δ_i in the underlying asset.

day 3 onward The agent recalculates Δ_i, and depending on whether the market went up or down, buys or sells. The agent's repurchase of stock is $\Delta_i - \Delta_{i-1}$

In the absence of the agent, the price evolves by Brownian motion,

$$P_i = P_{i-1}\left(1 + r_f dt + \sigma\sqrt{dt}Z_i\right) \tag{6}$$

where

$$Z_i \sim N(0,1)$$

However, suppose that the set of agents in aggregate who engage in the trade is large. Therefore, as a group, they exerts a significant price impact with their purchases. The price impact is denoted by $\lambda > 0$. The price dynamics that take account of the impact of sales and purchases modifies (6) so that

$$P_i = P_{i-1}\left(1 + r_f dt + \sigma\sqrt{dt}Z_i + \lambda\left(\Delta_i - \Delta_{i-1}\right)\right) \tag{7}$$

In other words, the price change reflects the aggregate sales or purchases of the agents, which in turn is the change in the delta of the option.

So far, we have examined the analogous case to the simple tabular example examined in the previous section. Let us now introduce the feature that, after the initial price shock due to the sale or purchase, the market price reverts back to some fundamental value over time. Therefore, the impact of the agent's trading decision will slowly reverse. In the terminology of Kyle (1985), the market exhibits "resiliency"..

Assume that the price impact of sales and purchases is slowly reversed, at a constant rate, over F days. Therefore, at each future day fraction $1/F$ of the initial shock $(\Delta_i - \Delta_{i-1})$ reverts. On any given day, price impacts over many previous days of trading are reverting. We denote the amount that reverts every day by H_i, so

$$H_i = \frac{1}{F} \sum_{j=1}^{F} (\Delta_{i-j} - \Delta_{i-j-1})$$

Therefore, we need to modify (7) to take this into account, and get

$$P_i = P_{i-1} \left(1 + r_f dt + \sigma\sqrt{dt} Z_i + \lambda (\Delta_i - \Delta_{i-1}) - \lambda H_i\right) \tag{8}$$

This is the form of the price function that we will now examine in our numerical exercise. The pricing function (8) differs from the simple tabular example above in two ways. First, the strike price of the option being hedged depends on the current price, rather than being fixed. Second, the price impact of trades is only temporary, and eventually the market reverts back to fundamental value.

3.1 Simulation

In the simulations reported below, we set the annual volatility at $\sigma = 0.25$, the number of time periods at $N = 30$, the annual risk free rate at $r_f = 0.05$. The number of days for the price impact to revert was set at $F = 30$. The strike price fraction is $\phi = 0.9$, and finally, the price impact factor is set at $\lambda = 0.25$.

We report two different types of results below. First we present a plot of a sample price path, where we fix the realization of the shocks and compare the price paths for $\lambda = 0.25$ and $\lambda = 0$. That is, we compare the price paths with and without the pricing impact of trades. We repeat the exercise four times, i.e., $Q = 4$. Since $Q \times N = 90$ the simulated price path tracks 90 days trading. The two price paths are shown in Figure 5.

In order to better understand the distributional properties of the model where the agent has a significant price impact, along with the resiliency, we

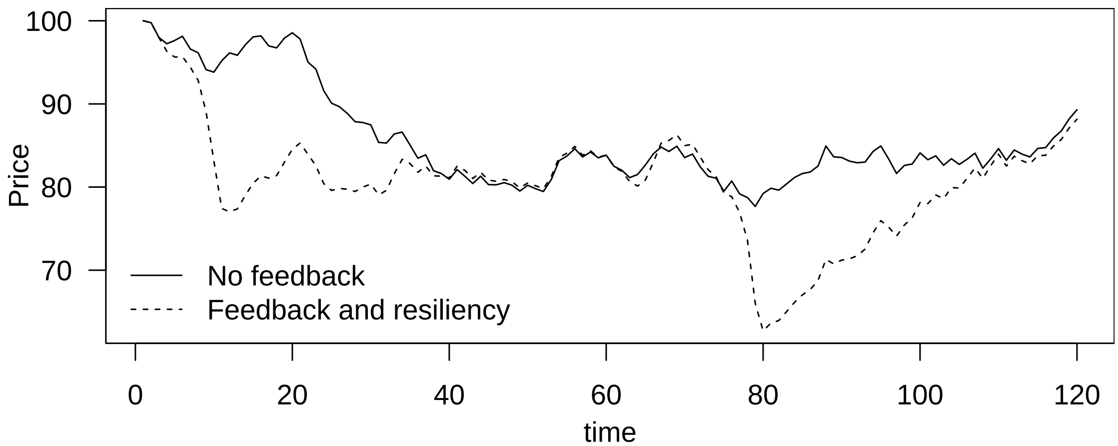

Figure 5. **Effect of endogenous shocks on price path.** This figure illustrates the price paths with and without the price impact of trades for the same realization of fundamentals shocks. The dotted line is for $\lambda = 0.25$ with reversion to fundamentals, and the solid line is the case for $\lambda = 0$.

Table 3. **Volatility and Kurtosis.** This table gives the volatility and kurtosis of the simulated densities where one has price feedback while the other does not. Both volatility and kurtosis increase substantially with feedback.

Model	Volatility	Kurtosis
No feedback	1.3	3.0
Resiliency	2.2	15.4

also repeat this $Q = 2,000$ times, to get the sample of size $Q \times N = 60,000$ days. These results are reported in Figure 6 and Table 3.

The contrasting price paths in Figure 5 demonstrate the considerable impact of price feedback on the dynamic path of prices. For some of the time, the two paths track each other closely, implying that the feedback effect of trading does not exert much effect. However, following large price moves, the two paths can diverge quite drastically. Such divergence is confirmation that the price feedback effects studied in the simple tabular example above can be shown to exert considerable influence in a more realistic dynamic setting. Notice also from Figure 5 that even with market resiliency, the price path with endogenous shocks can stray quite far from the fundamental value.

The endogenous price paths leave their mark on the distribution of returns, also. In particular, the shape of the density exhibit the typical “fat

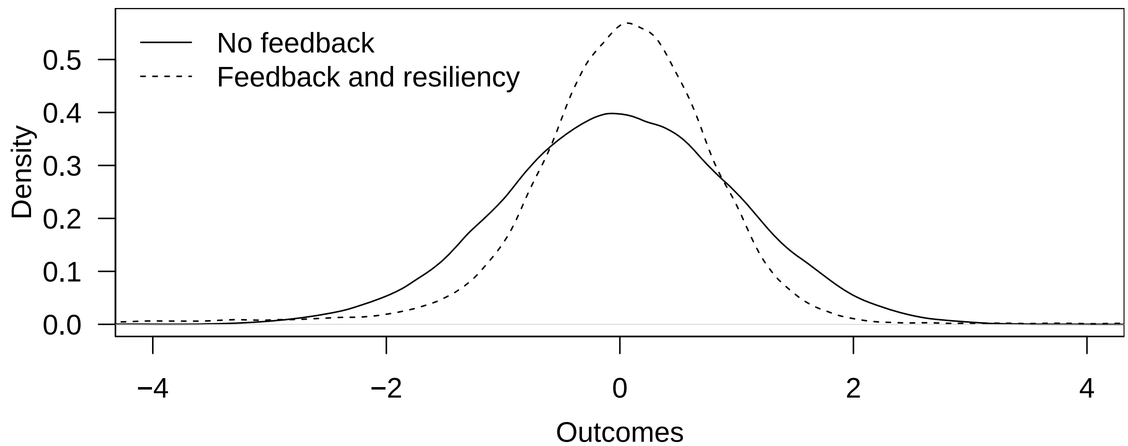

Figure 6. **Return density with price feedback.** This figure compares the return density with price feedback (dotted line) and without price feedback (solid line).

tail" shape relative to the normal density. Figure 6 presents the simulated density with endogenous feedback (dotted line) relative to the Gaussian fundamentals-driven returns given by the solid line. We see the typical tell-tale signs of a more sharply peaked distribution of returns with more extreme outcomes in the tails. Table 3 confirms both the greater volatility and the substantially higher kurtosis when the returns incorporate endogenous feedback. The kurtosis goes from 3 to 15.4 as we introduce endogenous feedback in prices, while volatility goes from 1.3 to 2.2. We see clearly the effect of feedback. Dynamic trading strategies coupled with endogenous risk increase overall market risk, whether measured by volatility or kurtosis.

4 Concluding Remarks

This paper has illustrated the possibility of endogenously generated extreme outcomes when prices play the dual role of both reflecting the underlyng fundamentals, but also driving the constraints on economic agents' actions. The illustration has relied on one type of constraint - the automatic response to buy and sell that results from the delta hedging strategy. We close the paper with some remarks on how our general approach can be usefully employed to address a wider range of market dynamics that rest on the same spirit of the dual nature of prices.

In Danielsson, Shin and Zigrand (2010), we show how the dual role of prices can amplify market risk itself and thereby drive the leverage of financial intermediaries. The depletion of bank capital and subsequent deleveraging by banks has been a central theme in the discussion of the recent global financial crisis and its impact on the real economy. Banks maintain enough equity to meet perceived risks arising from shocks to the value of their asset holdings. However, such realized risks should itself be considered endogenous, and depend on the ability of banks to take on risky exposures. When the banking sector suffers depletion of capital due to losses on its assets, its capacity to take on risky exposures diminishes as the risk constraint tightens. In other words, balance sheet capacity, risk constraints and market risk premiums should all be determined simultaneously in equilibrium.

Danielsson, Shin and Zigrand (2010) show that it is possible to solve for the equilibrium in closed form in a dynamic banking model and examine how balance sheet capacity, volatility and risk premiums are jointly determined. One key feature of the equilibrium is that risk premiums are high when banking sector capital is depleted, implying that projects that previously received funding from the banking sector no longer do so with depleted capital. This is a result that is reminiscent of a "credit crunch" due to banking sector losses, and follows from the following confluence of forces. Banks are risk neutral but their capacity to take on risky exposures is limited by their capital cushion. As their capital is depleted, their risk constraints bind harder, and their behavior resembles that of risk-averse investors. Indeed, the Lagrange multiplier associated with the capital constraint enters into the banks' lending decisions just like a risk aversion parameter. As banks suffer erosion of their capital, equilibrium volatility increases at the same time as their "as if" risk aversion also increases. This combination of increasing risk and increased risk aversion leads to a rise in the risk premium in the economy. The expected returns to risky assets increase, and projects that previously received funding from the banking sector no longer receives funding.

The fact that risk premiums are determined by aggregate banking sector capital is very much in line with recent "macroprudential" thinking among policy makers whose aim is to ensure that banking sector stress tests are in place to ensure that the banking sector has sufficient capacity to perform its economic role of channeling funding from savers to borrowers. This is in contrast to the previously "microprudential" concern with ensuring that banks have sufficient capital to serve as a buffer against loss that protects depositors (and hence the deposit insurance agency) from losses. Whereas microprudential concerns have to do with avoiding fiscal costs (due to bank recapitalization), macroprudential concerns have to do with maintaining banking sector lending capacity.

More generally, the study of endogenous risk lies at the confluence of two strands in the literature. One strand is the literature on crisis dynamics in competitive equilibrium, such as Gennotte and Leland (1990), Geanakoplos (1997, 2009) and Geanakoplos and Zame (2003). The second strand is the corporate finance literature that draws insights on balance sheet constraints, such as Shleifer and Vishny's (1997) observation that margin constraints limit the ability of arbitrageurs to exploit price differences, as well as Holmström and Tirole's (1998) work on debt capacities.

The results in Danielsson, Shin and Zigrand (2010) tie together these two strands of the literature, and therefore share points of contact with a recent literature on balance sheet constraints enter as a channel of contagion. Kiyotaki and Moore (1997) and Gromb and Vayanos (2002) are early papers in this spirit. Brunnermeier and Pedersen (2009) emphasize the "margin spirals" that result where capital constraints set off amplified feedback effects. Garleanu and Pedersen (2009) extend the CAPM by incorporating a capital constraint to show how assets with the same fundamental risk may trade at different prices. He and Krishnamurthy (2007) have studied a dynamic asset pricing model with intermediaries, where the intermediaries' capital constraints enter into the asset pricing problem as a determinant of portfolio capacity. Amplification through wealth effects was studied by Xiong (2001), Kyle and Xiong (2001) who show that shocks to arbitrageur wealth can amplify volatility when the arbitrageurs react to price changes by rebalancing their portfolios.

These studies have focused on the financial market dynamics almost exclusively, rather than on the macroeconomic issues concerned with the impact of financial shocks on the real economy. The linking of financial dynamics driven by such constraints and the macroeconomics literature is an important task that would yield many important insights into business cycles.

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