Procyclical Leverage and Endogenous Risk*

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1 Introduction

To what extent do financial conditions fluctuate due to fluctuations in leverage? The "leverage effect" discussed by Black (1976) and Schwert (1989) is well known, but fluctuating leverage has received renewed attention in the aftermath of the financial crisis, with the focus being on the impact of fluctuating leverage on the capacity of banks to bear risks. Our contribution is to bring these two themes together into a unified treatment of leverage and volatility.

Leverage is procyclical for banks and other financial intermediaries - that is, leverage is high during booms and low during busts. Two strands of the recent literature have highlighted the link between procyclical leverage and financial stability. The first concerns endogenous determination of leverage when assets serve as collateral. Geanakoplos (1997, 2009) and Fostel and Geanakoplos (2008, 2012) have popularized the general equilibrium framework where assets serve as collateral, in which the risk bearing capacity of the financial system can be severely diminished following shocks to fundamentals.

The second relevant strand of the literature on leverage is the impact of procyclical leverage on systemic risk. Gorton (2007, 2009) and Gorton and Metrick (2010) have explored the analogy between classical bank runs where depositors withdraw their funds from conventional banks and the modern run in capital markets with securitized claims where runs are driven by the increased collateral requirements (increased "haircuts") and hence the reduced capacity to borrow.

Much of the recent literature has focused on the financial crisis, and less attention has been given to the link between leverage and volatility, even though the study of volatility has been central to discussions of leverage since the initial Black (1976) and Schwert (1989) contributions.

Our contribution is to complete the circle between leverage and volatility by endogenizing the procyclicality of leverage. Leverage and volatility are intimately linked, as the capacity of intermediaries to take on risk exposures

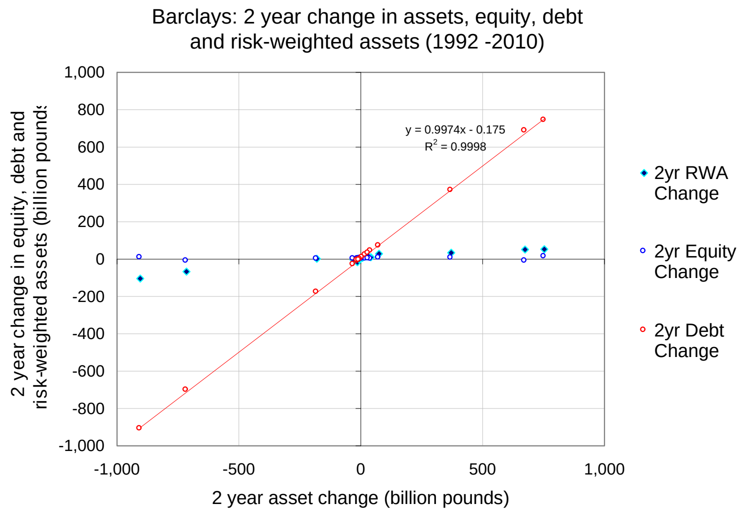

Figure 1: Scatter chart of relationship between the two year change in total assets of Barclays against two-year changes in debt, equity and risk-weighted assets (Source: Bankscope)

depends on the volatility of asset returns. However, equilibrium volatility is an endogenous variable and depends on the ability of intermediaries to take on risky exposures. So, we have a circularity. Solving for equilibrium entails solving simultaneously for volatility, risk premia and balance sheet capacity.

The circularity between procyclical leverage and volatility is encapsulated in Figure 1, which shows the scatter chart of the two-year changes in debt, equity and risk-weighted assets to changes in total assets of Barclays. The pattern in Figure 1 is typical of banks across countries and across business sectors.[1] More precisely, Figure 1 plots $\{(\Delta A_t, \Delta E_t)\}$, $\{(\Delta A_t, \Delta D_t)\}$ and $\{(\Delta A_t, \Delta \mathrm{RWA}_t)\}$ where ΔA_t is the two-year change in assets measured annually, and where ΔE_t, ΔD_t and $\Delta \mathrm{RWA}_t$ are the two-year changes in equity, debt, and risk-weighted assets, respectively.

The fitted line through $\{(\Delta A_t, \Delta D_t)\}$ has slope very close to 1, suggesting that assets expand or contract dollar for dollar (or pound for pound) through

[1] See Adrian and Shin (2010, 2011) and Adrian, Colla and Shin (2012).

a change in debt. What is especially notable is how the risk-weighted assets of the bank barely change, even as the raw assets change by large amounts. The fact that risk-weighted assets barely increase even as raw assets are increasing rapidly attests to the lowering of measured risks during upswings. However, the causation in the reverse direction will also be operating – that is, the compression of volatility is induced by the increase in credit supply. With such two-way causation, we need to solve for the equilibrium that fully encompasses the circularity between volatility and leverage.

Leverage is procyclical in Figure 1, as asset changes are driven by changes in debt, not equity. It is important to note that the leverage in Figure 1 is measured with respect to *book equity* - i.e. equity as implied by the bank's portfolio. Book equity is analogous to the haircut in a securitized borrowing transaction. In this sense, the procyclicality of leverage is another reflection of haircuts on collateral increasing in market downturns.

An alternative measure of equity would have been the bank's *market capitalization*, which gives the market price of its traded shares. However, since our interest is in the portfolio decision of the bank, book equity is the appropriate notion of equity for our purpose. It is important to remember that market capitalization is not the same as the marked-to-market value of the book equity in this context. Market capitalization reflects market discount rates for future cash flows, as well as the snapshot value of the bank's portfolio.

In this paper, we construct a dynamic banking model where the banking sector consists of risk-neutral banks that are subject to a Value-at-Risk (VaR) constraint that requires them to maintain a capital cushion that limits their probability of insolvency at all times to some known constant. We solve for the equilibrium in closed form and examine how bank lending, volatility and risk premia are jointly determined.

Our model and its solution have a number of attractive features. First, we are able to solve our model in closed form, where equilibrium volatility, risk premia and leverage can all be solved as functions of a single state variable θ, with

$$\theta = \frac{\text{Size of long-only sector}}{\text{Banking sector equity}} \tag{1}$$

where the numerator is some constant times the holding of the risky asset by the non-bank sector. The denominator is the aggregate capital of the banking sector as a whole. In terms of our state variable θ, the equilibrium volatility $\sigma(\theta)$ takes the particularly simple form:

$$\sigma(\theta) = \left\{ \begin{array}{c} \text{fundamental} \\ \text{volatility} \end{array} \right\} \times \theta \exp\{-\theta\} \times F(\theta) \tag{2}$$

where "fundamental volatility" refers to the volatility due to the exogenous shocks to the economy, and the function $F(\theta)$ ensures that in the limiting case without a banking sector (i.e. as $\theta \to \infty$), the equilibrium volatility converges to the fundamental volatility. Thus, the shape of the volatility function $\sigma(\theta)$ is driven by the term $\theta \exp\{-\theta\}$. In particular, we show that there is a cut-off value of θ above which equilibrium volatility exceeds the fundamental volatility, but below which equilibrium volatility is below fundamental volatility. In equilibrium, aggregate bank leverage is inversely proportional to equilibrium volatility. In this way, we capture the notion that volatility is high and leverage low when bank capital is depleted, and thereby provide the link with Black (1976) and Schwert (1989), who first documented the empirical feature that declining asset prices lead to increased volatility.

The second attractive feature of our framework is that it is flexible enough to extend the analysis to the multi-asset setting. In the multi-asset version of our model, closed form solutions for volatilities, correlations and leverage are still available for some special cases. Leverage exhibits the same procyclical features as in the one dimensional case. We show that correlations in returns emerge endogenously even though the fundamentals driving the asset returns are independent, and that the correlation can be characterized quite cleanly in terms of the fundamentals. Indeed, the closed form solution is sufficiently compact that we can address more applied topics such as derivatives pricing and the shape of the volatility curve. Kim and Kon (1994), Tauchen, Zhang and Liu (1996) and Anderson, Bollerslev, Diebold and Ebens (2001) find that the leverage effect is stronger for stock indices than for individual securities. Our framework provides a compact explanation of the phenomenon.

Our paper adds to recent work where balance sheet constraints enter as a channel of contagion. Amplification through wealth effects was studied by Xiong (2001) who showed that shocks to arbitrageur wealth can amplify volatility when the arbitrageurs react to price changes by rebalancing their portfolios. However, Xiong (2001) assumes portfolio investors with log preferences, which leads to leverage that is *countercyclical* - that is, leverage is high during busts and low during booms. Log preferences have also been important in more recent contributions, such as He and Krishnamurthy (2010, 2012) and Brunnermeier and Sannikov (2010, 2011). Given the importance of procyclical leverage in the portfolio chosen by banks and other intermediaries, we take a different tack in this paper.

The spirit of our equilibrium construction is closer to the recent paper by Adrian and Boyarchenko (2012), who model the procyclical fluctuations in leverage through risk constraints. The paper by Adrian and Boyarchenko (2012) as well as ours hark back to an earlier strand of the literature on Lagrange multiplier associated with Value-at-risk constraints. An earlier paper of ours (Danielsson, Shin and Zigrand (2004)) had backward-looking learning rather than solving for equilibrium in a rational expectations model. Brunnermeier and Pedersen (2009) and Oehmke (2008) have explored the consequences of fluctuating Lagrange multipliers associated with balance sheet constraints, but without solving for the fixed point problem.

Relative to these earlier papers, our contribution is to solve for the equilibrium returns, volatility, correlations and leverage in closed form, and to show how the simplicity of the solution allows the multiple asset extension. The tractability afforded by our closed-form solution is instrumental in deriving several of the insights in our paper, and opens up a number of useful avenues to link the banking literature with insights from asset pricing.

We begin with a general statement of the problem and introduce our closed-form solution. We then extend the analysis to the general multi-asset case where co-movements can be explicitly studied. We conclude by outlining some implications of our analysis for the analysis of financial crises and for macroeconomics more generally.

2 The Model

Our model describes the interactions between two groups of investors - passive investors and active investors. The passive investors can be thought of as value investors such as households, pension funds and mutual funds, while the active investors can be interpreted as banks and other intermediaries.

The risky securities can be interpreted as loans granted to ultimate borrowers, but where there is a risk that the borrowers do not fully repay the loan. Figure 2 depicts the relationships. Under this interpretation, the market value of the risky securities can be thought of as the marked-to-market value of loans granted to the ultimate borrowers. The value investors' holding of the risky security can be interpreted as the credit that is granted *directly* by the household sector (through the holding of corporate bonds, for example), while the holding of the risky securities by the active investors can be given the interpretation of *intermediated credit* through the banking sector.

Let time be indexed by $t \in [0, \infty)$. There are $N > 0$ non-dividend paying risky assets as well as a risk-free bond We will focus later on the case where $N = 1$, but we state the problem for the general N asset case. The price of the ith risky asset at date t is denoted P_t^i. We will look for an equilibrium in which the price processes for the risky assets follow:

$$\frac{dP_t^i}{P_t^i} = \mu_t^i dt + \sigma_t^i dW_t \qquad ; i = 1, \dots, N \tag{3}$$

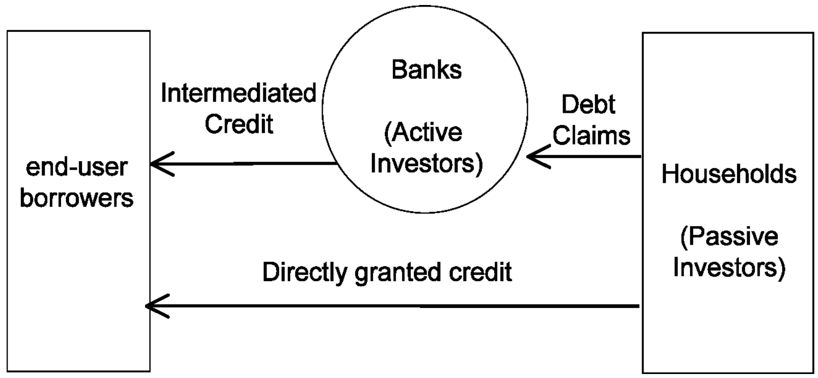

Figure 2: Intermediated and Directly Granted Credit

where W_t is an $N \times 1$ vector of independent Brownian motions, and where the scalar μ_t^i and the $1 \times N$ vector σ_t^i are as yet undetermined processes that will be solved in equilibrium. The risk-free bond has price B_t at date t, which is given by $B_0 = 1$ and $dB_t = rB_t dt$, where r is constant.

2.1 Portfolio Choice of Banks

The banks (the financial intermediaries, or "FIs") have short horizons and maximize the instantaneous expected returns on their loan portfolio subject to a Value-at-Risk constraint where its capital V is required to be sufficiently large to cover Value-at-Risk. We use "capital" and "equity" interchangeably in what follows.

We do not provide microfoundations for the VaR rule here,[2] but capital budgeting practices based on measured risks (such as VaR) are well-established among banks, and we adopt it here as a key feature of our model. The short-horizon nature of our model is admittedly stark, but can be seen as reflecting the same types of frictions that give rise to the use of con-

[2]See Adrian and Shin (2011) for one possible microfoundation in a contracting model with moral hazard, and Danielsson and Zigrand (2008) for a forward looking general equilibrium model with production where a VaR constraint reduces the probability of a systemic event caused by a free-riding externality during the refinancing stage.

straints such as VaR, and other commonly observed institutional features among banks and other large financial institutions. Finally, note that we have denoted the bank's capital as V without a subscript for the bank, as it will turn out that there is a natural aggregate result where only the aggregate banking sector capital matters for equilibrium, rather than the distribution of bank capital.

Let a_t^i be the number of units of the ith risky asset held at date t, and denote the dollar amount invested in risky security i by

$$D_t^i := a_t^i P_t^i \tag{4}$$

The budget constraint of the trader is

$$b_t B_t = V_t - a_t^\top P_t = V_t - \sum_{i=1}^{N} D_t^i \tag{5}$$

where V_t is the trader's capital and where $x^\top$ is the transpose of x. The dynamic budget constraint governs the evolution of capitalas follows:

$$\begin{aligned} dV_t &= a_t^\top dP_t + b_t dB_t \\ &= \left[rV_t + D_t^\top (\mu_t - r) \right] dt + D_t^\top \sigma_t dW_t \end{aligned} \tag{6}$$

where $D^\top$ denotes the transpose of D, and where σ_t is the $N \times N$ diffusion matrix, row i of which is σ_t^i. In (6), we have abused notation slightly by writing $r = (r, \ldots, r)$ in order to reduce notational clutter. The context should make it clear where r is the scalar or the vector.

From (6), the expected capital gain is

$$E_t[dV_t] = [rV_t + D_t^\top (\mu_t - r)]dt \tag{7}$$

and the variance of the trader's equity is

$$\mathrm{Var}_t(dV_t) = D_t^\top \sigma_t \sigma_t^\top D_t dt \tag{8}$$

We denote the variance-covariance matrix of instantaneous returns as $\Sigma_t := \sigma_t \sigma_t^\top$. The bank is risk-neutral, and maximizes return (7) subject to its

Value-at-Risk constraint, which can be written as some positive constant α times the forward-looking standard deviation of returns on the bank's equity. We take the bank's equity V_t as the state variable . Assuming that the bank is solvent (i.e. $V_t > 0$), the bank's maximization problem can be written as[3]:

$$\max_{D_t} \; rV_t + D_t^\top(\mu_t - r) \qquad \text{subject to} \qquad \alpha\sqrt{D_t^\top \sigma_t \sigma_t^\top D_t} \leq V_t \tag{9}$$

Once the dollar values $\{D_t^i\}_{i=1}^N$ of the risky assets are determined, the bank's value of debt is determined by the balance sheet identity:

$$b_t B_t = V_t - \sum\nolimits_i D_t^i \tag{10}$$

The first-order condition for the optimal D is

$$\mu_t - r = \alpha(D_t^\top \Sigma_t D_t)^{-1/2} \gamma_t \Sigma_t D_t \tag{11}$$

where γ_t is the Lagrange multiplier associated with the VaR constraint.

Hence,

$$D_t = \frac{1}{\alpha(D_t^\top \Sigma_t D_t)^{-1/2}\gamma_t} \Sigma_t^{-1}(\mu_t - r) \tag{12}$$

When $\mu_t \neq r$, as will occur in equilibrium, the objective function is monotonic in D_t by risk-neutrality, and the constraint must bind. Hence,

$$V_t = \alpha\sqrt{D_t^\top \Sigma_t D_t} \tag{13}$$

and therefore

$$D_t = \frac{V_t}{\alpha^2 \gamma_t} \Sigma_t^{-1}(\mu_t - r) \tag{14}$$

Notice that the optimal portfolio is similar to the mean-variance optimal portfolio allocation, where the Lagrange multiplier γ_t appears in the denominator, just like a risk-aversion coefficient. We thus have a foretaste of the

[3]In Appendix A, we show more rigorously why the Value-at-Risk of the bank can be written as α times the instantaneous standard deviation of the return on equity, $\sqrt{D_t^\top \sigma_t \sigma_t^\top D_t}$.

main theme of the paper - namely, that the banks in our model are risk-neutral, but they will behave like risk averse investors whose risk aversion appears to shift in line with the Lagrange multiplier γ. Substituting into (13) and rearranging we have

$$\gamma_t = \frac{\sqrt{\xi_t}}{\alpha} \tag{15}$$

where

$$\xi_t := (\mu_t - r)^\top \Sigma_t^{-1} (\mu_t - r) \geq 0 \tag{16}$$

The Lagrange multiplier γ_t for the VaR constraint is thus proportional to the generalized Sharpe ratio $\sqrt{\xi}$ for the risky assets in the economy. Although traders are risk-neutral, the VaR constraint makes them act as if they were risk-averse with a coefficient of relative risk-aversion of $\alpha^2 \gamma_t = \alpha \sqrt{\xi_t}$. As α becomes small, the VaR constraint binds less and banks' willingness to take on risk increases.

Notice that the Lagrange multiplier γ_t does not depend directly on equity V_t. Intuitively, an additional unit of capital relaxes the VaR constraint by a multiple α of standard deviation, leading to an increase in the expected return equal to a multiple α of the generalized Sharpe ratio, i.e. the risk-premium on the portfolio per unit of standard deviation. This should not depend on V_t directly, and indeed we can verify this fact from (16).

Finally, we can solve for the risky asset holdings as

$$D_t = \frac{V_t}{\alpha \sqrt{\xi_t}} \Sigma_t^{-1} (\mu_t - r) \tag{17}$$

The optimal holding of risky assets is homogeneous of degree one in equity V_t. This simplifies our analysis greatly, and allows us to solve for a closed form solution for the equilibrium. Also, the fact that the Lagrange multiplier depends only on market-wide features and not on individual capital levels simplifies our task of aggregation across traders and allows us to view demand (17) without loss of generality as the aggregate demand by the FI sector with aggregate capital of V_t.

2.2 Closing the Model with Value Investors

We close the model by introducing value investors who supply downward-sloping demand curves for the risky assets. The slope of the value investors' demand curves will determine the size of the price feedback effect. Suppose that the value investors in aggregate have the following vector-valued exogenous demand schedule for the risky assets, $y_t = (y_t^1, \ldots, y_t^N)$ where

$$y_t = \Sigma_t^{-1} \begin{bmatrix} \delta^1 \left(z_t^1 - \ln P_t^1\right) \\ \vdots \\ \delta^N \left(z_t^N - \ln P_t^N\right) \end{bmatrix} \tag{18}$$

where P_t^i is the market price for risky asset i and where dz_t^i is a (favorable) Itô demand shock to the demand of asset i (or a unfavorable supply shock to security i) to be specified further. Each demand curve can be viewed as a downward sloping demand hit by demand shocks, with δ^i being a scaling parameter that determines the slope of the demand curve. The particular form adopted for these exogenous demands is to aid tractability of the equilibrium pricing function, as we will see shortly. We can interpret these demands as coming from risk averse value investors who wish to hold a portfolio of the risky securities where their holding depends on the expected upside return, $\ln(P_t^{*i}/P_t^i)$, relative to the private values or benchmark prices P_t^{*i} which are given by $e^{z_t^i}$ (i.e. benchmark prices P_t^{*i} correspond to the equilibrium prices that would obtain in a fundamental economy without a financial sector). The coefficients δ play the role of risk tolerance parameters.

Bringing together the demands of the banks and the value investors, the market-clearing condition $D_t + y_t = 0$ can be written as

$$\frac{V_t}{\alpha\sqrt{\xi_t}}(\mu_t - r) + \begin{bmatrix} \delta^1 \left(z_t^1 - \ln P_t^1\right) \\ \vdots \\ \delta^N \left(z_t^N - \ln P_t^N\right) \end{bmatrix} = 0 \tag{19}$$

Equilibrium prices are therefore

$$P_t^i = \exp\left(\frac{V_t}{\alpha\delta^i\sqrt{\xi_t}}(\mu_t^i - r) + z_t^i\right); \qquad i = 1, \ldots, N \tag{20}$$

In solving for the rational expectations equilibrium (REE) of our model, our strategy is to begin with some exogenous stochastic process that drives the passive traders' demands for the risky assets (the fundamental "seeds" of the model, so to speak), and then solve for the endogenously generated stochastic process that governs the prices of the risky assets.

In particular, we will look for an equilibrium in which the price processes for the risky assets are of the form:

$$\frac{dP_t^i}{P_t^i} = \mu_t^i dt + \sigma_t^i dW_t \qquad ; i = 1, \ldots, N \tag{21}$$

where W_t is an $N \times 1$ vector of independent Brownian motions, and where the scalar μ_t^i and $1 \times N$ vector σ_t^i are as yet undetermined coefficients that will be solved in equilibrium. The "seeds" of uncertainty in the equilibrium model are given by the demand shocks of the value investors:

$$dz_t^i = r^* dt + \eta\sigma_z^i dW_t \tag{22}$$

where σ_z^i is a $1 \times N$ vector that governs which Brownian shocks will get impounded into the demand shocks and therefore govern the correlation structure of the demand shocks. We assume that the stacked $N \times N$ matrix σ_z is of full rank and that $r^* > r$, so that demand shocks reflect risk aversion of the value investors.

Our focus is on the way that the (endogenous) diffusion terms $\{\sigma_t^i\}$ of the return process depends on the (exogenous) shock terms $\{\sigma_z^i\}$, and how the exogenous noise terms may be amplified in equilibrium via the risk constraints of the active traders. Indeed, we will see that the relationship between the two sets of diffusions generate a rich set of empirical predictions.

3 Equilibrium with Single Risky Asset

Before examining the general problem with N risky assets, we first solve the case of with single risky asset. We will look for an equilibrium where the price of the risky asset follows the process:

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dW_t \tag{23}$$

where μ_t and σ_t are, as yet, undetermined coefficients to be solved in equilibrium, and W_t is a standard scalar Brownian motion. The "seeds" of uncertainty in the model are given by the exogenous demand shocks to the value investors' demands:

$$dz_t = r^* dt + \eta \sigma_z dW_t \tag{24}$$

where $\sigma_z > 0$ and $\eta > 0$ are known constants. For the single risky asset case, note that

$$\xi_t = \frac{(\mu_t - r)^2}{\sigma_t^2} \tag{25}$$

Substituting into (20), and confining our attention to regions where the Sharpe ratio $\frac{\mu_t - r}{\sigma_t}$ is strictly positive, we can write the price of the risky asset as

$$P_t = \exp\left(z_t + \frac{\sigma_t V_t}{\alpha\delta}\right) \tag{26}$$

From (23) we have, by hypothesis,

$$d\ln P_t = \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t \tag{27}$$

Meanwhile, taking the log of (26) and applying Itô's Lemma gives

$$\begin{aligned}
d\ln P_t &= d\left(z_t + \frac{\sigma_t V_t}{\alpha\delta}\right) \\
&= r^* dt + \eta\sigma_z dW_t + \frac{1}{\alpha\delta} d(\sigma_t V_t) \\
&= r^* dt + \eta\sigma_z dW_t + \frac{1}{\alpha\delta}\left(\sigma_t dV_t + V_t d\sigma_t + dV_t d\sigma_t\right) \qquad (28)
\end{aligned}$$

Now use Itô's Lemma on $\sigma(V_t)$:

$$\begin{aligned} d\sigma_t &= \frac{\partial\sigma}{\partial V_t} dV_t + \frac{1}{2}\frac{\partial^2\sigma}{\partial (V_t)^2}(dV_t)^2 \\ &= \left\{ \frac{\partial\sigma}{\partial V_t}\left[rV_t + \frac{V_t(\mu_t - r)}{\alpha\sigma_t} \right] + \frac{1}{2}\frac{\partial^2\sigma}{\partial (V_t)^2}\left(\frac{V_t}{\alpha}\right)^2 \right\} dt + \frac{\partial\sigma}{\partial V_t}\frac{V_t}{\alpha} dW_t \end{aligned} \tag{29}$$

where (29) follows from

$$\begin{aligned} dV_t &= [rV_t + D_t(\mu_t - r)]dt + D_t\sigma_t dW_t \\ &= \left[rV_t + \frac{V_t(\mu_t - r)}{\alpha\sigma_t} \right] dt + \frac{V_t}{\alpha} dW_t \end{aligned} \tag{30}$$

and the fact that $D_t = \frac{V_t}{\alpha\sigma_t}$ due to the binding VaR constraint. Defining the aggregate (gross) *bank leverage ratio* by $\ell_t := \frac{D_t}{V_t}$, we see that $\ell_t = \frac{1}{\alpha\sigma_t}$ from which the procyclical properties of ℓ follow once we have characterised the equilibrium σ.

Notice also that $(dV_t)^2 = \left(\frac{V_t}{\alpha}\right)^2 dt$. We thus obtain diffusion equations for V_t and for σ_t itself.

Substituting back into (28) and regrouping all dt terms into a new drift term:

$$d\ln P_t = (\text{drift term})\ dt + \left[\eta\sigma_z + \frac{1}{\alpha\delta}\left(\sigma_t\frac{V_t}{\alpha} + V_t\frac{\partial\sigma_t}{\partial V_t}\frac{V_t}{\alpha} \right) \right] dW_t \tag{31}$$

We can solve for the equilibrium diffusion σ_t by comparing coefficients between (31) and (27). We have an equation for the equilibrium diffusion given by:

$$\sigma(V_t) = \eta\sigma_z + \frac{1}{\alpha\delta}\left(\sigma_t\frac{V_t}{\alpha} + V_t\frac{\partial\sigma_t}{\partial V_t}\frac{V_t}{\alpha} \right) \tag{32}$$

which can be written as the ordinary differential equation (ODE):

$$V_t^2\frac{\partial\sigma}{\partial V_t} = \alpha^2\delta(\sigma_t - \eta\sigma_z) - V_t\sigma_t \tag{33}$$

It can be verified by differentiation that the generic solution to this ODE is given by

$$\sigma(V_t) = \frac{1}{V_t}e^{-\frac{\alpha^2\delta}{V_t}}\left[c - \alpha^2\delta\eta\sigma_z\int_{-\frac{\alpha^2\delta}{V_t}}^{\infty}\frac{e^{-u}}{u}du \right] \tag{34}$$

where c is a constant of integration.

We can set $c = 0$ through the following natural restriction in our model. The only randomness in our economy stems from the shocks to the value investor demands. If we let $\delta \to 0$, value investors' demand goes to zero and we get the REE $\sigma(V_t) = \frac{c}{V_t}$. Since the limit economy should be non-random, we require that returns also are riskless, $\sigma(V_t) = 0$, implying that $c = 0$.

We thus obtain a unique closed form solution to the rational expectations equilibrium for the single risky asset case. Setting $c = 0$ and simplifying, we arrive at the following succinct closed form solution

$$\sigma(V_t) = \eta\sigma_z \frac{\alpha^2\delta}{V_t} \exp\left\{-\frac{\alpha^2\delta}{V_t}\right\} \times \mathrm{Ei}\left(\frac{\alpha^2\delta}{V_t}\right) \tag{35}$$

where $\mathrm{Ei}\,(w)$ is the well-known[4] exponential integral function:

$$\mathrm{Ei}\,(w) \equiv -\int_{-w}^{\infty} \frac{e^{-u}}{u} du \tag{36}$$

The $\mathrm{Ei}\,(w)$ function is defined provided $w \neq 0$. The expression $\alpha^2\delta/V_t$ which appears prominently in the closed form solution (35) can be interpreted as the relative scale or size of the value investor sector (parameter δ) compared to the banking sector (total capital V_t normalized by VaR).

To bring out the structure of the solution better, let θ be the relative size of the long-only sector relative to the banking sector by defining θ as

$$\theta_t \equiv \frac{\alpha^2\delta}{V_t} \tag{37}$$

Then, the closed form solution for equilibrium volatility can be written as a function of θ_t as:

$$\sigma(\theta_t) = \eta\sigma_z\theta_t \exp\{-\theta_t\} \times \mathrm{Ei}\,(\theta_t) \tag{38}$$

The closed form solution also reveals much about the basic shape of the volatility function $\sigma\,(\theta_t)$. Consider the limiting case when the banking sector

[4] See http://mathworld.wolfram.com/ExponentialIntegral.html

is very small, that is, $\theta_t \to \infty$. Although θ_t becomes large, the exponential term $\exp\{-\theta_t\}$ in (38) goes to zero much faster, and so the product $\theta_t \exp\{-\theta_t\}$ becomes small. However, since we have exogenous shocks to the value investor demands, there should still be non-zero volatility at the limit, given by the fundamental volatility $\eta\sigma_z$. The role of the exponential integral term $\mathrm{Ei}(\theta_t)$ is to ensure that the limiting volatility when the banking sector becoms small is given by the volatility that would hold in the absence of a banking sector.

The procyclical properties of leverage, being higher in good – meaning well-capitalised – times and lower during bank crises then follow directly from

$$\ell_t := \frac{D_t}{V_t} = \frac{1}{\alpha\sigma(\theta_t)}$$

The equilibrium risk premium in our model is given by the drift μ_t (the expected instantaneous return on the risky asset) which can be solved in closed form, and is given by

$$\mu_t = r + \frac{\sigma_t}{2\alpha\eta\sigma_z}\left\{2\alpha(r^* - r) + \alpha\sigma_t^2 - \eta\sigma_z + (\sigma_t - \eta\sigma_z)\left[2\alpha^2 r + \frac{\alpha^2\delta}{V_t} - 2\right]\right\} \tag{39}$$

We can see that μ_t depends on the diffusion σ_t, so that when the expression in the square brackets is positive, μ_t is increasing in σ_t. Thus, even though banks are risk-neutral, they are prevented by their VaR constraint from fully exploiting all positive expected return opportunities. The larger is σ_t, the tighter is the risk constraint, and hence the higher is the expected return μ_t. Note that the expression in the square brackets is positive when V_t is small, which is consistent with the VaR constraint binding more tightly.

Also, notice that as the VaR constraint becomes tighter, $\lim_{\alpha\to\infty}\sigma_t = \eta\sigma_z$ and $\lim_{\alpha\to\infty}\alpha(\sigma_t - \eta\sigma_z) = 0$ so that in the limit we have $\mu_t - \frac{(\eta\sigma_z)^2}{2} = r^*$, confirming our interpretation of r^* as the value investor sector's benchmark log-return.

The information contained in the risk premium μ_t and its relationship

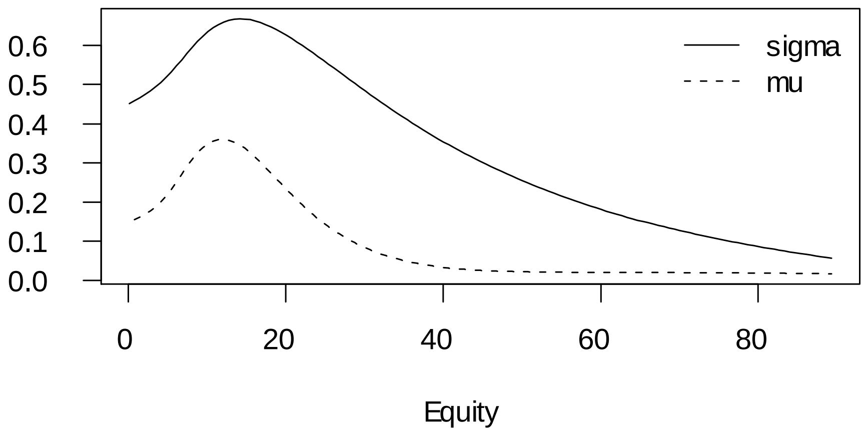

Figure 3: Risk Premium and Volatility as Functions of Bank Equity

with the diffusion σ_t can be summarized alternatively in terms of the Sharpe ratio, which can be written as

$$\frac{\mu_t - r}{\sigma_t} = \frac{1}{2\alpha\eta\sigma_z}\left\{2\alpha(r^* - r) + \alpha\sigma_t^2 - \eta\sigma_z + (\sigma_t - \eta\sigma_z)\left[2\alpha^2 r + \frac{\alpha^2\delta}{V_t} - 2\right]\right\} \tag{40}$$

The countercyclical shape of the Sharpe Ratio follows directly from the shape of the diffusion coefficient σ_t.

3.1 Numerical Example

We illustrate the properties of our closed form solution by means of a numerical example. Figure 3 plots the equilibrium diffusion σ_t and the drift μ_t as a function of the state variable V_t. The parameters chosen for this plot were $r = 0.01$, $r^* = 0.047$, $\delta = 6$, $\alpha = 2.7$, $\sigma_z = 0.3$, $\eta = 1.5$

As suggested by the closed form solution (35), the plot of σ_t is non-monotonic, with a peak when V_t is low. Also, note that when $V = 0$, we have $\sigma = 0.45$, which is the fundamental volatility given by the product of σ_z and η ($= 0.3 \times 1.5$). This non-monotonic shape of the volatility function

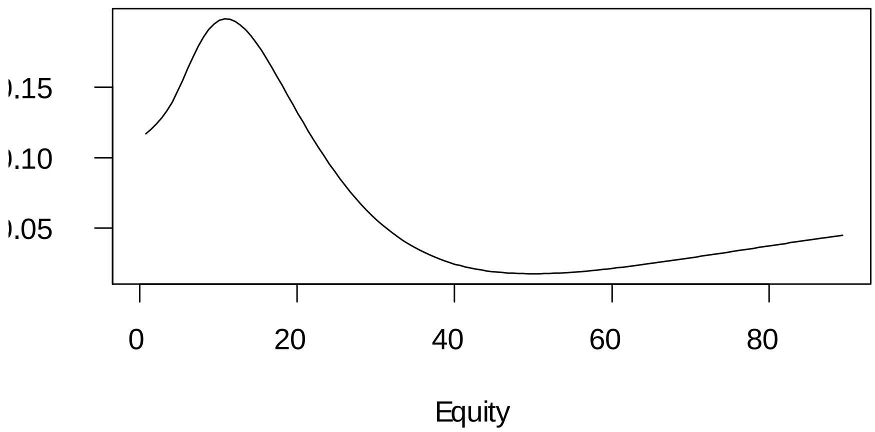

Figure 4: Lagrange Multiplier of Bank Capital Constraint

is completely general, and does not depend on the parameters chosen. We provide further arguments in the appendix.[5]

What Figure 3 reveals is that the feedback effect generating endogenous volatility is strongest for low to intermediate values of V_t. This is so, since there are two countervailing effects. If V_t is very small - close to zero, say - then there is very little impact of the banks' portfolio decision on the price of the security since the VaR constraint restricts the banks' risky holdings to be negligible. Therefore, both σ_t and μ_t are small. At the opposite extreme, if V_t is very large, then banks begin to act more and more like an unconstrained trader. Since the trader is risk–neutral, the expected drift μ_t is pushed down to the risk–free rate, and the volatility σ_t declines.

However, at an intermediate level of V_t, the feedback effect is maximized, where a positive price shock leads to greater purchases, which raises prices further, which leads to greater purchases, and so on. This feedback effect increases the equilibrium volatility σ_t. Due to the risk constraint, the risk-neutral banks behave "as if" they were risk averse, and the equilibrium drift μ_t reflects this feature of the model. The risk premium μ_t rises with σ_t, since

[5] See Mele (2007) for a discussion of the stylized facts, and for a model generating countercyclical statistics in a more standard framework.

both risk and risk aversion increase as bank equity is depleted.

Indeed, as we have commented already, the Lagrange multiplier associated with the risk constraint is the Sharpe ratio in this simple one asset context. The closed-form expression for the Lagrangian is plotted in Figure 4. We see that the Sharpe ratio rises and falls roughly the same pattern with σ_t and μ_t. However, the notable feature of Figure 4 is that the Lagrange multiplier may actually start increasing again when V is large. This is because the Lagrange multiplier reflects the bank's return on equity (ROE), and ROE is affected by the degree of leverage taken on by the bank. When V becomes large, the volatility falls so that bank leverage increases. What Figure 4 shows is that the increased leverage may start to come into play for large values of V.

Figure 5 gives scatter charts for the relationship between the asset growth and leverage for four sample realizations of the model. Each scatter chart for a particular sample path is accompanied by the price series for that sample path.'

The scatter charts reveal the characteristic clustering of dots around the 45-degree line, as shown by Adrian and Shin (2010) for the Wall Street investment banks. The notable feature from the scatter charts is how the slope and degree of clustering depends on the price realizations. When the price path is low, many of the observations are for the upward-sloping part of the volatility function $\sigma(V)$. Along the upward-sloping part, equity depletion associated with price declines is accompanied by a decline in Value-at-Risk, and hence an uptick in leverage. These observations are those below the 45-degree line, but where leverage goes up. However, when the realizations are mainly those on the downward-sloping part of the volatility curve, the scatter chart hugs more closely the 45-degree line. The second panel in Figure 5 shows this best.

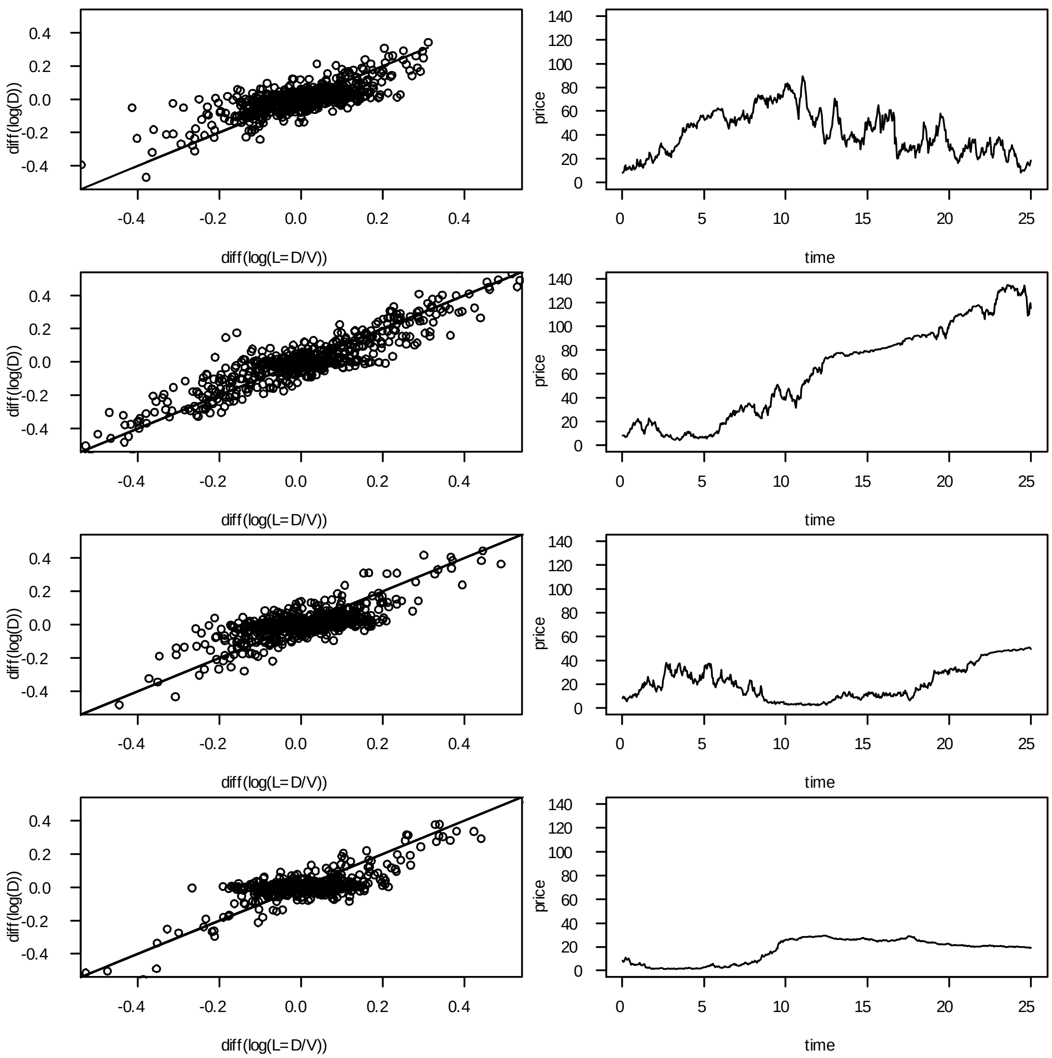

Figure 5: Scatter Charts of Asset and Leverage Changes

4 Equilibrium with Many Risky Assets

4.1 General Specification

We now turn to the case with $N > 1$ risky assets and look for an equilibrium in which the prices of risky assets follow:

$$\frac{dP_t^i}{P_t^i} = \mu_t^i dt + \sigma_t^i dW_t \tag{41}$$

where W_t is an $N \times 1$ vector of independent Brownian motions, and where μ_t^i and σ_t^i are terms to be solved in equilibrium. The demand shocks of the passive traders are given by

$$dz_t^i = r^* dt + \eta \sigma_z^i dW_t \tag{42}$$

where σ_z^i is a $1 \times N$ vector that governs which Brownian shocks affect the passive traders' demands.

We denote conjectured quantities with a tilde. For instance, conjectured drift and diffusion terms are $\tilde{\mu}, \tilde{\sigma}$ respectively and the actual drift and diffusions are μ and σ respectively. For notational convenience, we define the scaled reward-to-risk factor

$$\lambda_t := \frac{1}{\sqrt{\xi_t}} \Sigma_t^{-1} (\mu_t - r) \tag{43}$$

Also, we use the following shorthands:

$$\beta_t^i := \frac{1}{\sqrt{\xi_t}} (\mu_t^i - r) \tag{44}$$

$$\epsilon_t^i := \frac{1}{\alpha^2 \delta^i} \beta_t^i + \frac{V_t}{\alpha^2 \delta^i} \frac{\partial \beta_t^i}{\partial V_t} \tag{45}$$

and where

$$\frac{\partial \epsilon_t^i}{\partial V_t} = \frac{2}{\alpha^2 \delta^i} \frac{\partial \beta_t^i}{\partial V_t} + \frac{V_t}{\alpha^2 \delta^i} \frac{\partial^2 \beta_t^i}{\partial V_t^2}$$

Under some conditions to be verified, we can compute the *actual* drift and diffusion terms of dP_t^i/P_t^i as a function of the conjectured drift and diffusion terms. By Itô's Lemma applied to (20) we have:

$$\sigma_t^i = \tilde{\epsilon}_t^i V_t \tilde{\lambda}_t^\top \tilde{\sigma}_t + \eta \sigma_z^i \tag{46}$$

We denote the $N \times 1$ vector of ones by $\mathbf{1}_N$, and the operator that replaces the main diagonal of the identity matrix by the vector v by $\mathrm{Diag}(v)$. Also, for simplicity we write r for $r\mathbf{1}_N$. Then we can stack the drifts into the vector μ_t, the diffusion coefficients into a matrix σ_t, etc.

We can solve the fixed point problem by specifying a beliefs updating process $(\tilde{\mu}, \tilde{\sigma})$ that when entered into the right hand side of the equation, generates the true return dynamics. In other words, we solve the fixed point problem by solving for self-fulfilling beliefs $(\tilde{\mu}_t, \tilde{\sigma}_t)$ in the equation:

$$\begin{bmatrix} \tilde{\mu}_t \\ \tilde{\sigma}_t \end{bmatrix} = \begin{bmatrix} \mu_t(\tilde{\mu}_t, \tilde{\sigma}_t) \\ \sigma_t(\tilde{\mu}_t, \tilde{\sigma}_t) \end{bmatrix}. \tag{47}$$

By stacking into a diffusion matrix, at a REE the diffusion matrix satisfies

$$\sigma_t = V_t \epsilon_t \lambda_t^\top \sigma_t + \eta \sigma_z \tag{48}$$

Using the fact that $\lambda_t^\top \sigma_t \sigma_t^\top = \beta_t^\top$, σ_t satisfies the following matrix quadratic equation $\sigma_t \sigma_t^\top = \eta \sigma_z \sigma_t^\top + V_t \epsilon_t \beta_t^\top$ so that

$$(\sigma_t - \eta \sigma_z)\sigma_t^\top = V_t \epsilon_t \beta_t^\top \tag{49}$$

The return diffusion in equilibrium is equal to the fundamental diffusion $\eta\sigma_z$ – the one occurring with no active FIs in the market – perturbed by an additional low-rank term that incorporates the rational equilibrium effects of the FIs on prices. Therefore, we have a decomposition of the diffusion matrix into that part which is due to the fundamentals of the economy, and the part which is due to the endogenous amplification that results from the actions of the active traders. The decomposition stems from relation (46) (keeping in mind that $\frac{V_t}{\alpha}\lambda_t\sigma_t$ equals the diffusion term of equity)

$$\sigma_t^i = \underbrace{\left(\frac{1}{\alpha\delta^i}\beta_t^i\right)}_{\substack{\text{feedback effect on vol}\\ \text{from VaR}}} (\text{vol of capital}) + \underbrace{\left(\frac{V}{\alpha\delta^i}\frac{\partial\beta_t^i}{\partial V_t}\right)}_{\substack{\text{feedback effect on vol}\\ \text{from changing expectations}}} (\text{vol of capital}) + \eta\sigma_z^i$$

We now solve for a representation of σ_t. Solutions to quadratic matrix equations can rarely be guaranteed to exist, much less being guaranteed to be computable in closed form. We provide a representation of the solution, should a solution exist. This solution diffusion matrix can be shown to be nonsingular, guaranteeing endogenously complete markets by the second fundamental theorem of asset pricing.

Denote the scalar

$$e_t := 1 - V_t\lambda_t^\top\epsilon_t$$

It follows from the Sherman-Morrison theorem (Sherman and Morrison (1949)) that $e_t = \text{Det}\left[I - V_t\epsilon\lambda_t^\top\right]$ and that if (and only if) $e_t \neq 0$ (to be verified in equilibrium) we can represent the diffusion matrix:

$$\sigma_t = \eta\left[\frac{V_t}{1 - V_t\lambda_t^\top\epsilon_t}\epsilon_t\lambda_t^\top + I\right]\sigma_z \tag{50}$$

We then have the following result.

Proposition 1 *The REE diffusion matrix σ_t and the variance-covariance matrix Σ_t are non-singular, and*

$$\sigma_t^{-1} = \frac{1}{\eta}\sigma_z^{-1}\left[I - V_t\epsilon_t\lambda_t^\top\right] \tag{51}$$

Proof. By the maintained assumption that σ_z is invertible, the lemma follows directly if we were able to show that $\left[\frac{V_t}{e_t}\epsilon_t\lambda_t^\top + I\right]$ is invertible. From the Sherman-Morrison theorem, this is true if $1+\frac{V_t}{e_t}\lambda_t^\top\epsilon_t \neq 0$, which simplifies to $1 \neq 0$. The expression for the inverse is the Sherman-Morrison formula. ∎

4.2 Closed Form Solution

To make further progress in the many asset case, we examine a special case that allows us to solve for the equilibrium in closed form. The special case allows us to reduce the dimensionality of the problem and utilize the ODE solution from the single risky asset case. Our focus here is on the correlation structure of the endogenous returns on the risky assets.

Assumption (Symmetry, S) The diffusion matrix for z is $\eta\tilde{\sigma}_z I_N$ where $\tilde{\sigma}_z > 0$ is a scalar and where I_N is the $N \times N$ identity matrix. Also, $\delta^i = \delta$ for all i.

The symmetry assumption enables us to solve the model in closed form and examine the changes in correlation. Together with the i.i.d. feature of the demand shocks we conjecture an REE where $\mu_t^i = \mu_t^1$, $\beta_t^i = \beta_t^1$, $\lambda_t^i = \lambda_t^1$, $\epsilon_t^i = \epsilon_t^1$, $\sigma_t^{ii} = \sigma_t^{11}$ and $\sigma_t^{ij} = \sigma_t^{12}$, $i \neq j$. First, notice that $\epsilon_t \lambda_t^\top = \epsilon_t^1 \lambda_t^1 \mathbf{1}\mathbf{1}^\top$, and that $\epsilon_t^\top \lambda_t = N\epsilon_t^1 \lambda_t^1$, where $\mathbf{1}$ is a $N \times 1$ vector of ones (so that $\mathbf{1}\mathbf{1}^\top$ is the $N \times N$ matrix with the number 1 everywhere).

From (50) we see that the diffusion matrix is given by

$$\eta\sigma_z \left(\frac{V_t \lambda_t^1 \epsilon_t^1}{1 - N V_t \lambda_t^1 \epsilon_t^1} \mathbf{1}\mathbf{1}^\top + I \right) \tag{52}$$

From here the benefit of symmetry becomes clear. At an REE we only need to solve for one diffusion variable, $\sigma_t^{ii} = \sigma_t^{11}$, since for $i \neq j$ the cross effects $\sigma_t^{ij} = \sigma_t^{12} = \sigma_t^{11} - \eta\tilde{\sigma}_z$ are then determined as well. Recall that σ_t^{ij} is the measure of the effect of a change in the demand shock of the jth security on the price of the ith security, and not the covariance. In other words, it governs the comovements between securities that would otherwise be independent. Define by $x_t \equiv x(V_t)$ the solution to the ODE (33) with η replaced by $\frac{\eta}{N}$, i.e. x_t is equal to the right-hand-side of (35) with η replaced by $\frac{\eta}{N}$. The proof of the following proposition is in the appendix.

Proposition 2 *Assume (S). The following is an REE.*

The REE diffusion coefficients are $\sigma_t^{ii} = x_t + \frac{N-1}{N}\eta\tilde{\sigma}_z$*, and for* $i \neq j$, $\sigma_t^{ij} = x_t - \frac{1}{N}\eta\tilde{\sigma}_z$*. Also,* $\Sigma_t^{ii} = Var_t(\text{return on security } i) = \eta^2\tilde{\sigma}_z^2 + \frac{1}{N}\left(N^2x_t^2 - \eta^2\tilde{\sigma}_z^2\right)$*, and for* $i \neq j$, $\Sigma_t^{ij} = Cov_t(\text{return on security } i, \ \text{return on security } j) = \frac{1}{N}\left(N^2x_t^2 - \eta^2\tilde{\sigma}_z^2\right)$ *and* $Corr_t(\text{return on security } i, \ \text{return on security } j) = \frac{Nx_t^2 - \frac{1}{N}\eta^2\tilde{\sigma}_z^2}{Nx_t^2 + \frac{N-1}{N}\eta^2\tilde{\sigma}_z^2}$.

Risky holdings are $D_t^i = \frac{V_t}{\alpha N^{3/2} x_t}$*. The gross leverage ratio is* $\ell_t := \frac{\sum_{i=1}^N D_t^i}{V_t} = \frac{1}{\alpha x_t \sqrt{N}}$.

The risk-reward relationship is given by

$$\frac{\mu_t^i - r}{x_t} = \frac{1}{2\alpha\frac{\eta}{N}\tilde{\sigma}_z}\left\{2\alpha(r^* - r) + \alpha\left(Nx_t^2 + \eta^2\tilde{\sigma}_z^2\frac{N-1}{N}\right) - \frac{\eta}{\sqrt{N}}\tilde{\sigma}_z + \sqrt{N}\left(x_t - \frac{\eta}{N}\tilde{\sigma}_z\right)\left[2\alpha^2 r + \frac{\alpha^2\delta}{V_t} - 2\right]\right\} \tag{53}$$

The intuition and form of the drift term is very similar to the $N = 1$ case and reduces to it if N is set equal to 1. Similarly, the leverage ratio remains procyclical and behaves like in the $N = 1$ case.

With multiple securities and with active banks, each idiosyncratic shock is transmitted through the system through the banks' portfolio decisions. This can be seen also from the fact that price i can be written as $P_t^i = \exp\left(R(\theta_t) + z_t^i\right)$, with z_t^i the idiosyncratic shock and $R(\theta_t) := \frac{\alpha\sigma(\theta_t)}{\theta_t\sqrt{N}}$ the aggregate shock. On the one hand this means that less than the full impact of the shock on security i will be transmitted into the asset return i, potentially leading to a less volatile return. The reason is that a smaller fraction of the asset portfolio is invested in asset i, reducing the extent of the feedback effect. On the other hand, the demand shocks to assets other than i will be impounded into return i, potentially leading to a more volatile return, depending on the extent of mutual cancellations due to the diversification effect on the FIs' equity. In a world with multiple risky securities satisfying the assumptions in the proposition, the extent of contagion across securities is given by $\sigma_t^{ij} = x_t - \frac{1}{N}\eta\tilde{\sigma}_z$, for $i \neq j$. In the absence of FIs, $x_t = x(0) = \frac{1}{N}\eta\tilde{\sigma}_z$, so any given security return is unaffected by the idiosyncratic shocks hitting other securities.

For comparison purposes, denote the scalar diffusion coefficient from the $N = 1$ case, as given by (34), by $\sigma_t^{N=1}$. The first direct effect can be characterized as follows: $\sigma_t^{11} < \sigma_t^{N=1}$ iff $\eta\tilde{\sigma}_z < \sigma_t^{N=1}$. In words, each security return is affected less by its own noise term than in a setting with only this one security, for small levels of capital. The reason for this latter effect lies in the fact that any given amount of FI capital needs to be allocated across multiple securities now. For capital levels larger than the critical level $V^* : \sigma_t^{N=1}(V^*) = \eta\tilde{\sigma}_z$, the direct effect is larger than in the $N = 1$ economy[6] because the (now less constrained) risk-neutral FIs tend to absorb aggregate return risk as opposed to idiosyncratic return risk. Whereas all uncertainty vanishes in the $N = 1$ case since FIs insure the residual demand when capital becomes plentiful ($\lim_{V\to\infty} \sigma_t^{N=1} = 0$), with $N > 1$ on the other hand individual volatility remains ($\lim_{V\to\infty} \Sigma_t^{11} = \frac{N-1}{N}\eta^2\tilde{\sigma}_z^2 > 0$) but the fact that correlations tend to -1 means that $\lim_{V\to\infty}$ Var(return on the equilibrium portfolio)$= 0$. So again as FI capital increases, aggregate equilibrium return uncertainty is washed out, even though returns continue to have idiosyncratic noise.

Combining direct and indirect effects, return variance is lower in the multi security case if V is small: $\Sigma_t^{11} < (\sigma_t^{N=1})^2$ iff $\eta^2\tilde{\sigma}_z^2/N^2 < x_t^2$. Still, as in the $N = 1$ case securities returns are more volatile with active banks ($V_t > 0$), provided capital is not too large.

Diversification across the N i.i.d. demand shocks lessens the feedback effect on prices to some extent. Since the VaR constraints bind hard for small levels of capital, the fact that idiosyncratic shocks are mixed and affect all securities implies that asset returns become more correlated for small capital levels. FIs tend to raise covariances by allowing the i.i.d. shocks that affect security i to be also affecting security $j \neq i$ through their portfolio choices. This effect has some similarities to the wealth effect on portfolio

[6] For instance, as $V \to \infty$, we have $\lim_{V\to\infty} \sigma_t^{11} = \frac{N-1}{N}\eta\tilde{\sigma}_z > 0 = \lim_{V\to\infty} \sigma_t^{N=1}$.

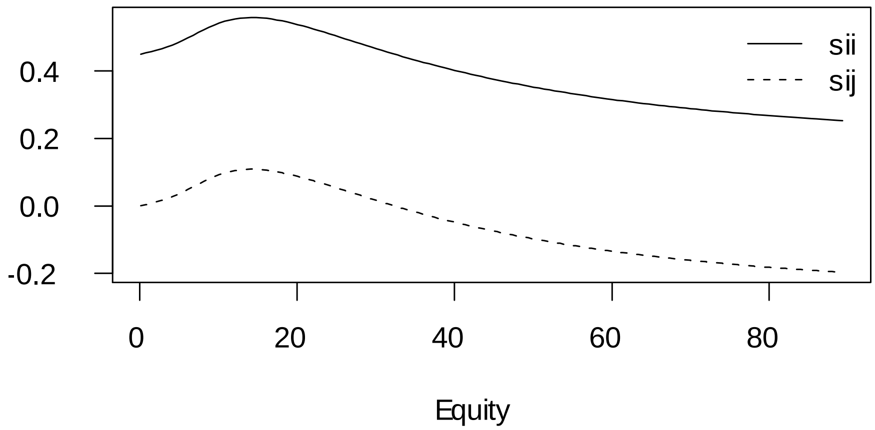

Figure 6: Cross Contagion across Risky Assets

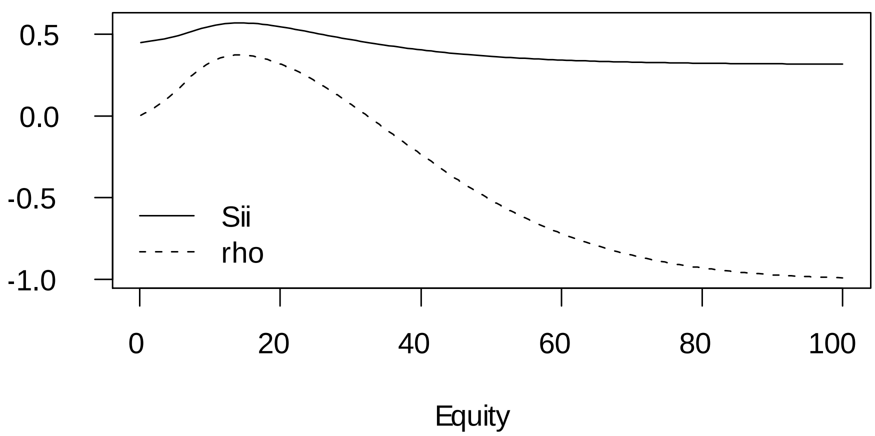

Figure 7: Return Correlations across Risky Assets

choice described by Kyle and Xiong (2001). The intuition is as follows. Without FIs, returns on all securities are independent. With a binding VaR constraint, in the face of losses, FIs' risk appetite decreases and they are forced to scale down the risk they have on their books. This leads to joint downward pressure on all risky securities.

This effect is indeed confirmed in an REE, leading to positively correlated returns. This effect is consistent with anecdotal evidence on the loss of diversification benefits suffered by hedge funds and other traders who rely on correlation patterns, when traders are hit by market shocks. The argument also works in reverse: as FIs start from a tiny capital basis that does not allow them to be much of a player and accumulate more capital, they are eager to purchase high Sharpe ratio securities. This joint buying tends to raise prices in tandem.

Figure 7 shows the correlation as a function of V. As can be seen on Figure 7, variances move together, and so do variances with correlations. This echoes the findings in Andersen et al (2001) who show that

> "there is a systematic tendency for the variances to move together, and for the correlations among the different stocks to be high/low when the variances for the underlying stocks are high/low, and when the correlations among the other stocks are also high/low."

They conjecture that these co-movements occur in a manner broadly consistent with a latent factor structure (the x process in our model).

5 Further Results

The logic of the feedback effects that underlies the shapes of the volatility, risk premia and Sharpe Ratio graphs naturally has a number of powerful corollaries that tie in with empirical regularities in the financial markets.

5.1 Leverage Effect

The "leverage effect" refers to the empirical regularity noted by Black (1976) and Schwert (1989) that declining asset prices lead to increased future volatility. Recent work by Kim and Kon (1994), Tauchen, Zhang and Liu (1996) and Anderson, Bollerslev, Diebold and Ebens (2001) find that the leverage effect is stronger for indices than for individual securities. This has been considered as a puzzle for a literal interpretation of the leverage effect, and we are not aware of theoretical explanations for this asymmetry. Our model also finds that the overall market volatility reacts more to falls than individual securities, and our story provides a natural intuition for the effect. As equity V is reduced, prices fall and volatilities increase. Since correlations also increase as equity falls, the volatility of the market portfolio increases more than the volatility of the individual securities underlying the market portfolio. Define $\tilde{V}$ so that $\frac{\partial x_t}{\partial V_t}(\tilde{V}) = 0$, meaning that the region where equity satisfies $V > \tilde{V}$ corresponds to the usual region right of the hump where capital losses lead to more volatile returns.

Proposition 3 *Assume $N > 1$. A decrease in V raises the volatility of the market more than it raises the volatility of an individual constituent security iff $V_t > \tilde{V}$.*

Proof. In our model it can easily be verified that the variance of the market portfolio is equal to $\Sigma_t^m := Nx_t^2$, and that $\Sigma_t^{ii} = Nx_t^2 + \frac{N-1}{N}\eta^2\tilde{\sigma}_z^2$. A few manipulations verify

$$\frac{\partial\sqrt{\Sigma_t^m}}{\partial V} - \frac{\partial\sqrt{\Sigma_t^{ii}}}{\partial V} = Nx_t\frac{\partial x_t}{\partial V}\left[\frac{1}{\sqrt{\Sigma_t^m}} - \frac{1}{\sqrt{\Sigma_t^{ii}}}\right] < 0$$

iff $V_t > \tilde{V}$, since $\Sigma_t^m < \Sigma_t^{ii}$ if $N > 1$.

5.2 Derivatives Pricing Implications

It is well known that the Black-Scholes-Merton implied volatilities exhibit a negative skew in moneyness K/S that is fading with longer time to maturity

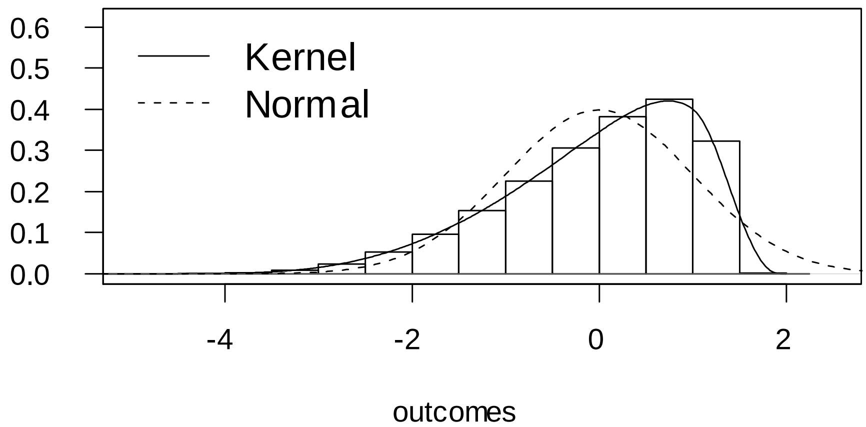

Figure 8: Skewed Density over Outcomes

(see for instance Aït-Sahalia and Lo (1998) for a formal econometric analysis). The usual intuition for the relative over-pricing of out-of-the-money (OTM) puts compared to OTM calls within the Black-Scholes-Merton model relies on the fact that OTM puts offer valuable protection against downside "pain points," and that such a downside either is expected to occur more frequently than similar upside movements or at least occurs in more volatile environments than would a similar upside movement, or that investors are willing to pay more to protect the downside compared to Black-Scholes. These effects imply a fatter left tail of the risk-neutral returns distribution compared to the Gaussian Black-Scholes model, as can be verified on Figure 8.

While a judicious choice of parameters allows stochastic volatility and local volatility models to generate implied volatility surfaces that exhibit the observed stylised facts, these models are mere representations and do not provide a rationale for the resulting skew surfaces. Our model provides a simple micro-founded channel through which the observed volatility skew is generated. In our framework the REE volatility function $|\sigma_t|$ largely depends negatively on bank capital V_t (except for very small values of V_t). Capital being random, volatility is stochastic. Since the value of the underlying risky

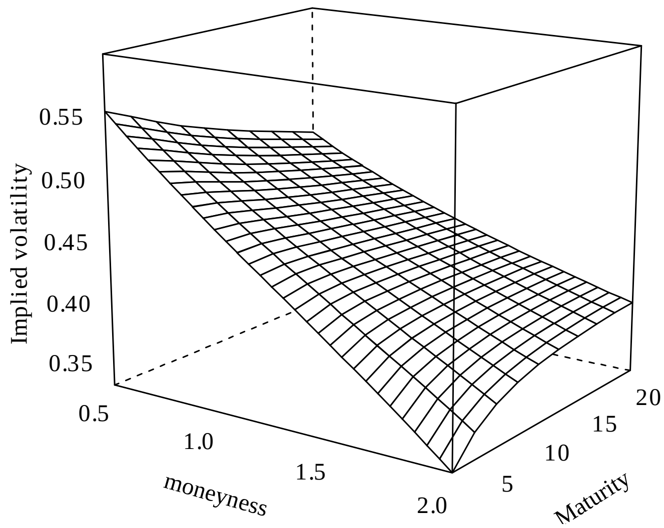

Figure 9: Implied Volatility Surface

asset (viewed as the overall market index) depends positively on bank capital over a large range of capital levels, but its volatility depends mostly negatively on bank capital, one can expect that the option generated implied volatility skew appears in equilibrium. Indeed, the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp\left(-\frac{1}{2}\int_0^T \left(\frac{\mu_t - r}{\sigma_t}\right)^2 dt - \int_0^T \frac{\mu_t - r}{\sigma_t} dW_t\right)$ allocates more mass to the left tail with the model's countercyclical market price of risk process $\frac{\mu_t - r}{\sigma_t}$ than with a constant market price of risk process, thereby generating the skew.

Figure 9 gives the implied volatility surface arising from our model in (K/S, maturity) space. We see the skew for each maturity, as well as a flattening over longer maturities. The flattening is due to the fact that over a longer horizon bank equity will more likely than not have drifted upwards and further out of the danger zone.

Our simple specification not only characterises the shape of the IV surface, it also provides predictions as to the dynamic evolution of these surfaces over the cycle. If one focuses on the at-the-money (ATM, moneyness of one) across various capital levels, one sees that the ATM implied vols (which would in this model be equivalent to the VIX index or similar) are counter-cyclical. An economy with higher capital levels has a lower VIX, and worsening economic circumstances lead to a higher VIX. This is a well-established empirical fact, so much so that the VIX is also referred to as the "investor fear gauge."

6 Concluding Remarks

The financial crisis of 2007-9 has served as a reminder of the important role played by the fluctuations in the leverage of banks and other financial intermediaries in driving financial conditions. Our contribution has been to provide a unified framework that links the procyclical nature of leverage with the classical literature on the volatility of market outcomes and the "leverage effect" of Black (1976) and Schwert (1989).

The work of Gorton (2007, 2009) and Geanakoplos (2010, 2011) has brought home to researchers and policy makers the importance of understanding the fluctuations in leverage in driving financial conditions and macroeconomic conditions more generally. For instance, Geanakoplos (2010, 2011) shows how thinking about haircuts pushes us to inquire into the leverage on the *new, marginal loan* when considering financial conditions, rather than the leverage on the existing book of assets. As such, financial conditions can be seen to have begun deteriorating in late 2006 and early 2007, even though leverage measured on aggregate assets remained high until 2009.

The procyclicality of leverage distinguishes our paper from most of the existing literature on financial shock amplifications that assume log preferences of investors. Log preferences imply that leverage is *countercyclical* - that is, leverage is high during busts, and low during booms. To the extent that our focus is on the investor's portfolio decision, the leverage should be measured with respect to the equity that is implied by the investor's portfolio. Hence, book equity is the appropriate notion when measuring leverage embedded in portfolio choice, not market capitalization. Thus, obtaining procyclical leverage should be a feature that researchers should have at the top of their list as a modeling objective. Procyclicality of leverage is simply the reflection of haircuts rising when market conditions deteriorate, as emphasized by Gorton (2007, 2009) and Gorton and Metrick (2010).

Procyclicality of leverage also distinguishes our paper from the "financial frictions" literature in macroeconomics that posits a fixed haircut, and hence a fixed leverage constraint which transmits shocks purely through shocks to net worth, rather than changes in leverage. Adrian, Colla and Shin (2012) argue that understanding the fluctuations in leverage has important consequences for modeling macroeconomic fluctuations.

Our objective has been to bring the recent discussion on financial crises back to the classical literature on volatility and the "leverage effect". Comparatively little attention has been given to the link between leverage and volatility in the financial crisis literature, even though the study of volatility has been central to discussions of leverage since the initial Black (1976) and Schwert (1989) contributions.

Our contribution has been to complete the circle between leverage and volatility by endogenizing the procyclicality of leverage. As we have seen, leverage and volatility are intimately linked, as the capacity of intermediaries to take on risk exposures depends on the volatility of asset returns. At the same time, equilibrium volatility is an endogenous variable and depends on the ability of intermediaries to take on risky exposures. We have addressed this circularity. Our contribution has been to address this circularity in a simple, closed-form solution with applications in a variety of areas. The issues of policy responses to the dangers of procyclicality would be one fruitful avenue to explore. Since equilibrium volatility is driven by the state variable θ_t measuring the relative size of the long-only sector to the banking capitalisation, a countercyclical capital adequacy policy that adjusts the restrictiveness of the VaR constraint to the cycle via $\alpha^2(V_t) = \text{constant}\frac{V_t}{\delta}$ would fix θ_t and therefore $\sigma(\theta_t)$. The nefarious feedback loops in a downturn are softened as a result. More research should uncover further uses of our framework.

Appendix A

In this appendix we show that the Value-at-Risk constraint can be written as $\alpha\sqrt{D_t^\top \sigma_t \sigma_t^\top D_t} \leq V_t$ for positive constant α. By definition VaR is a quantile of a portfolio return. Standardising by $\sigma(h)$ the volatility of $V_{t+h} - V_t$, the VaR satisfies

$$\Pr_t\left(\frac{V_{t+h} - V_t}{\sigma(h)} \leq -\frac{\mathrm{VaR}(h)}{\sigma(h)}\right) = \epsilon$$

Given the normality of $V_{t+h} - V_t$ at h horizon for small h, and the negligible drift, $\Phi\left(-\frac{\mathrm{VaR}(h)}{\sigma(h)}\right) = \epsilon$ and so $\mathrm{VaR}(h) = [-\Phi^{-1}(\epsilon)]\sigma(h)$. The VaR-constraint says that the capital V underlying such a portfolio is sufficient, in that for some constant $c(h)$, we have $V_t \geq c(h)\mathrm{VaR}(h)$ at horizon h. For small h, $\mathrm{VaR}(h)$ is small, so that $c(h)$ must be decreasing in h. Since $\sigma(h) = \sqrt{h}\sqrt{D_t^\top \sigma_t \sigma_t D_t}$ for small h, we choose $c(h) = c/\sqrt{h}$ and so we can write the VaR constraint as

$$V_t \geq \underbrace{\left[c(-\Phi^{-1}(\epsilon))\right]}_{=\alpha}\sqrt{D_t^\top \sigma_t \sigma_t D_t}$$

which is the constraint written in the maximization problem (9).

Appendix B

In this appendix, we derive further formal properties of the diffusion term.

Lemma (Properties of the Diffusion Term)

[**S1**] $\lim_{V_t \to 0} \sigma(V_t) = \eta\sigma_z$
[**S2**] $\lim_{V_t \to \infty} \sigma(V_t) = 0$ and $\lim_{V_t \to \infty} V_t\sigma(V_t) = \infty$
[**S3**] $\lim_{V_t \to 0} \frac{\partial\sigma}{\partial V_t} = \frac{\eta\sigma_z}{\alpha^2\delta}$ and $\lim_{V_t \to 0} \frac{\partial^2\sigma}{\partial V_t^2} = \frac{4\eta\sigma_z}{(\alpha^2\delta)^2}$. [Call $f(V) := \frac{\sigma_t - \eta\sigma_z}{V_t}$ and notice that $\lim_{V \to 0} f(V) = \lim_{V \to 0} \frac{\partial\sigma}{\partial V}$. Since we know the expression for $\frac{\partial\sigma}{\partial V}$ by (33), we see that the problem can be transformed into $\lim f = \lim \frac{1}{V_t}[\alpha^2\delta f(V) - \sigma]$. In turn, we can replace $\frac{\sigma_t}{V_t}$ definitionally by $f + \frac{\eta\sigma_z}{V}$ to

get to $\lim f = \lim \frac{f(V)[\alpha^2\delta - V] - \eta\sigma_z}{V}$. If $\lim f$ is not equal to the constant given here, then the RHS diverges. Since the denominator of the RHS converges to zero, so must the numerator. Thus the constant is the one shown here. The proof of the second limit is similar.]

[**S4**] $\{V^* \in \mathbb{R} : \sigma(V^*) = 0\}$ is a singleton. At V^*, σ is strictly decreasing. [The second observation comes from (33) while the first one comes from the fact that the mapping $V \mapsto \int_{-\frac{\alpha^2\delta}{V}}^{\infty} \frac{e^{-u}}{u} du$ is a bijection between $\mathbb{R}_+$ and $\mathbb{R}$, so for each chosen c, there is a unique $V(c)$ setting $\left[c - \alpha^2\delta\eta\sigma_z \int_{-\frac{\alpha^2\delta}{V(c)}}^{\infty} \frac{e^{-u}}{u} du\right] = 0$.]

[**S5**] $\sigma(V)$ has exactly one minimum and one maximum. The minimum is at V' s.t. $\sigma(V') < 0$. The maximum is at V'' s.t. $\sigma(V'') > 0$.

Proof of Proposition 2 First, we can read off (52) the variables of interest as $\sigma_t^{ii} = \eta\tilde{\sigma}_z \left(\frac{V_t\epsilon_t\lambda_t}{1 - NV_t\epsilon_t\lambda_t} + 1\right)$ and $\sigma_t^{ij} = \sigma_t^{12} = \eta\tilde{\sigma}_z \frac{V_t\epsilon_t\lambda_t}{1 - NV_t\epsilon_t\lambda_t} = \sigma_t^{11} - \eta\tilde{\sigma}_z$.

Next, we compute the variance-covariance matrix, the square of the diffusion matrix (52):

$$\Sigma_t = \sigma_t\sigma_t = \eta^2\tilde{\sigma}_z^2[I_N + m_t\mathbf{1}\mathbf{1}^\top] = \eta^2\tilde{\sigma}_z^2 I_N + g_t\mathbf{1}\mathbf{1}^\top$$

where

$$\begin{aligned} m_t &:= N\left(\frac{V_t\epsilon_t^1\lambda_t^1}{1 - NV_t\epsilon_t^1\lambda_t^1}\right)^2 + 2\frac{V_t\epsilon_t^1\lambda_t^1}{1 - NV_t\epsilon_t^1\lambda_t^2} \\ &= \frac{1}{\eta^2\tilde{\sigma}_z^2}(\sigma_t^{11} - \eta\tilde{\sigma}_z)\left[2\eta\tilde{\sigma}_z + N(\sigma_t^{11} - \eta\tilde{\sigma}_z)\right] \\ g_t &: = m_t\eta^2\tilde{\sigma}_z^2 \end{aligned}$$

where we used the fact that $\eta\tilde{\sigma}_z \frac{V_t\epsilon_t^1\lambda_t^1}{1 - NV_t\epsilon_t^1\lambda_t^1} = \sigma_t^{11} - \eta\tilde{\sigma}_z$. Then insert Σ_t into the reward-to-risk equation $\Sigma_t\lambda_t^1\mathbf{1} = \frac{\mu_t^1 - r}{\sqrt{\xi}}\mathbf{1}$ to get $\sqrt{\xi_t}\lambda_t^1\left[\eta\tilde{\sigma}_z + N(\sigma_t^{11} - \eta\tilde{\sigma}_z)\right]^2 = \mu_t^1 - r$.

Next compute ξ_t. By definition, $\xi_t := (\mu_t^1 - r)^2\mathbf{1}^\top\Sigma_t^{-1}\mathbf{1}$. Since $1 + g_t(\eta\tilde{\sigma}_z)^{-2}N \neq 0$, by the Sherman-Morrison theorem we see that

$$\Sigma_t^{-1} = (\eta\tilde{\sigma}_z)^{-2}I - \frac{g_t}{(\eta\tilde{\sigma}_z)^4 + N(\eta\tilde{\sigma}_z)^2 g_t}\mathbf{1}\mathbf{1}^\top$$

and therefore that

$$\xi_t = (\mu_t^1 - r)^2 N(\eta\tilde{\sigma}_z)^{-2}\left[1 - \frac{Ng_t}{(\eta\tilde{\sigma}_z)^2 + Ng_t}\right]$$

Inserting the expression for ξ_t into the expression for λ_t^1 we get, using the fact that $[\eta\tilde{\sigma}_z + N(\sigma_t^{11} - \eta\tilde{\sigma}_z)]^2 = Ng_t + (\eta\tilde{\sigma}_z)^2$,

$$\lambda_t^1 = \frac{\iota_A \iota_B}{\sqrt{N}[\eta\tilde{\sigma}_z + N(\sigma_t^{11} - \eta\tilde{\sigma}_z)]}$$

where ι is the sign function, $A := \mu_t^1 - r$ and $B := N\sigma_t^{11} - (N-1)\eta\tilde{\sigma}_z$. Using again the fact that $[\eta\tilde{\sigma}_z + N(\sigma_t^{11} - \eta\tilde{\sigma}_z)]^2 = Ng_t + (\eta\tilde{\sigma}_z)^2$, we see that

$$\beta_t^1 = \iota_A \iota_B \frac{1}{\sqrt{N}}\left[\eta\tilde{\sigma}_z + N(\sigma_t^{11} - \eta\tilde{\sigma}_z)\right]$$

By definition of ϵ_t^1:

$$\epsilon_t^1 = \frac{1}{\alpha^2\delta}\iota_A \iota_B \left[\frac{1}{\sqrt{N}}[\eta\tilde{\sigma}_z + N(\sigma_t^{11} - \eta\tilde{\sigma}_z)] + V_t\sqrt{N}\frac{\partial \sigma_t^{11}}{\partial V_t}\right] \tag{54}$$

Inserting all these expressions into the equation for σ_t^{11}, $\sigma_t^{11} = \eta\tilde{\sigma}_z \frac{1-(N-1)V_t\lambda_t^1\epsilon_t^1}{1-NV_t\lambda_t^1\epsilon_t^1}$ and defining $x_t := \frac{1}{N}\eta\tilde{\sigma}_z + (\sigma_t^{11} - \eta\tilde{\sigma}_z)$, the resulting equation is the ODE (33) with η replaced by η/N and where $\sigma(V)$ is replaced by $x(V)$.

As to risky holdings, we know that $D_t = \frac{V_t}{\alpha}\lambda_t$. Noticing that $\eta\tilde{\sigma}_z + N(\sigma_t^{ii} - \eta\tilde{\sigma}_z) = Nx_t$, we find that $\lambda_t^1 = \frac{1}{N^{3/2}x_t}$ from which the expression for D_t follows.

Finally we compute the risk premia. Using Itô's Lemma on (20) we get

$$\mu_t^i - \frac{1}{2}\Sigma_t^{11} = \alpha\epsilon_t^1(\text{drift of } V_t) + r^* dt + \frac{1}{2}\alpha\frac{\partial \epsilon_t^1}{\partial V_t}(\text{diffusion of } V_t)^2$$

Now the drift of capital can be seen to be equal to

$$\text{drift of } V_t = rV_t + D_t^\top(\mu_t - r) = rV_t + (\mu_t^1 - r)\frac{V_t}{\alpha\sqrt{N}x_t}$$

and using $\sigma_t = \left(x_t - \frac{1}{N}\eta\tilde{\sigma}_z\right)\mathbf{1}\mathbf{1}^\top + \eta\tilde{\sigma}_z I$ the squared diffusion term can be verified to be equal to $\frac{V_t^2}{\alpha^2}$. The drift equation becomes

$$(\mu_t^1 - r)\left[1 - \epsilon_t^1 \frac{V_t}{\sqrt{N}x_t}\right] = \frac{1}{2}(\sigma_t^{11})^2 + \alpha^2\epsilon_t^1 r V_t + \frac{1}{2}\frac{V_t^2}{\alpha}\frac{\partial\epsilon_t^1}{\partial V_t}$$

We can rewrite (54) by inserting the ODE for x_t to get rid of the partial derivative term:

$$\epsilon_t^1 = \iota_A\iota_B\frac{\sqrt{N}}{V_t}\left(x_t - \frac{\eta}{N}\tilde{\sigma}_z\right)$$

Performing the differentiation of ϵ_t^1 and inserting into the drift equation completes the proof.

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